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CLASS GROUP RELATIONS IN CYCLOTOMIC FIELDS.[1]

By Robert E. MacKenzie.

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1. Introduction. Let F be the cyclotomic extension of the field R of rational numbers obtained by the adjunction of a primitive n-th root of unity ϵ, $n \geqq 3$.

Let N be the multiplicative group of residue classes of the rational integers prime to n modulo n. If q is an element of N, then q will be regarded as either a residue class modulo n or as a rational integer representing this residue class, whichever is appropriate to the context. For each $q \in N$, let σ_q be the uniquely determined automorphism of F over R for which $\sigma_q(\epsilon^q) = \epsilon$. By this definition N may be identified with the Galois group of F over R.

The principal result contained in this paper is the following theorem.

Theorem 4. *Let $\mathfrak{R}$ be any ideal class of F. For all rational integers s and t,*

$$\prod_{q \in N} \sigma_q(\mathfrak{R})^{[q(s+t)/n]-[qs/n]-[qt/n]}$$

is the principal class. ($[x]$ indicates the greatest integer $\leqq$ the rational number x.)

This theorem lists a set of relations among the elements of the class group of any cyclotomic extension of the rational field. In the present state of algebraic number theory it is a rather isolated fact. It would be of considerable interest to know whether it has a more intimate connection with class field theory.

Theorem 4 is a generalization of a known theorem [2] for the case where n is a prime. The key to the proof consists in regarding the n-th power residue symbol [3] $(\alpha/\mathfrak{p})$ as a function on the additive group of the residue class field of the prime $\mathfrak{p}$ and employing the relation between the convolution of the functions $(\alpha/\mathfrak{p})^s$ and $(\alpha/\mathfrak{p})^t$ and the product of their Fourier trans-

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[1] Received November 12, 1951.

[2] D. Hilbert, "Die Theorie der algebraischen Zahlkörper," *Jahresbericht der Deutschen Mathematiker Vereinigung*, vol. 4 (1897), § 109.

[3] H. Hasse, "Bericht über neurere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper," *ibid.*, Ergänzungsband VI (1930), § 10.

forms. Theorems 1 and 2 concern the factorization of the values of the Fourier transform of $(\alpha/\mathfrak{p})^s$. Theorem 3 gives the factorization of the values of the convolution of $(\alpha/\mathfrak{p})^s$ and $(\alpha/\mathfrak{p})^t$.

2. The Fourier transforms. Let $\mathfrak{p}$ be a prime ideal of the ring of integers of F which does not divide n. Let Φ be the additive group of the residue class field of $\mathfrak{p}$ and Φ^* the multiplicative group of non-zero residue classes. If α is an element of Φ (or Φ^*), then α will be regarded either as a residue class or as an integer which represents this residue class, whichever suits the context. $N\mathfrak{p}$ is the customary notation for the order of Φ.

Let X be the character group [4] of Φ^*. For $\chi \in X$ the Fourier transform ϕ_χ of χ is the function $\phi_\chi(\lambda) = \Sigma_\alpha \chi(\alpha)\lambda(\alpha)$, $\alpha \in \Phi^*$, defined on the characters [4] λ of Φ. Among the elements of X occur the functions $(\alpha/\mathfrak{p})^s$, where $(\alpha/\mathfrak{p})$ is the n-th power residue symbol [3] in F and s is a rational integer. When it requires special consideration, the Fourier transform of $(\alpha/\mathfrak{p})^s$ will be denoted by ϕ_s. For convenience, $(0/\mathfrak{p})$ will be set equal to 0. Then $(\alpha/\mathfrak{p})^s$ becomes a function defined on Φ.

THEOREM 1. *If* $\lambda \neq 1$, *then* $|\prod_{\chi \in X} \phi_\chi(\lambda)|^2 = N\mathfrak{p}^{N\mathfrak{p}-2}$.

Proof. Order the elements of Φ^* and of X in a fixed manner and consider the determinants

$$\Delta = \det\{\lambda(\alpha\beta) \mid \alpha, \beta \in \Phi^*\}$$

and

$$D = \det\{\chi(\alpha) \mid \chi \in X, \alpha \in \Phi^*\},$$

each consisting of $N\mathfrak{p} - 1$ rows and columns. Summing on α gives

$$D\Delta = \det\{\overline{\chi(\beta)}\ \Sigma_\alpha \chi(\alpha\beta)\lambda(\alpha\beta) \mid \chi \in X, \beta \in \Phi^*\},$$

where $\overline{\chi(\beta)}$ is the complex conjugate of $\chi(\beta)$. The sum occurring in this determinant is $\phi_\chi(\lambda)$ and hence $D\Delta = \bar{D}\Pi_\chi \phi_\chi(\lambda)$, $\chi \in X$.

D may be shown to be different from zero by forming

$$D\bar{D} = \det\{\Sigma_\chi \chi(\alpha)\overline{\chi(\beta)} \mid \alpha, \beta \in \Phi^*\}.$$

The sum occurring in this determinant is zero unless $\alpha = \beta$, in which case it is $N\mathfrak{p} - 1$. Therefore $D\bar{D} = (N\mathfrak{p} - 1)^{N\mathfrak{p}-1}$. It now follows that

$$|\Pi_\chi \phi_\chi(\lambda)| = |\Delta|.$$

[4] The values of the characters are roots of unity.

The theorem requires the evaluation of $|\Delta|^2$. Since

$$|\Delta| = |\det\{\lambda(-\nu\beta) \mid \nu, \beta \varepsilon \Phi^*\}|$$

it follows by suming on β that

$$|\Delta|^2 = |\det\{\Sigma_\beta \lambda(\{\alpha - \nu\}\beta) \mid \alpha, \nu \varepsilon \Phi^*\}|.$$

The sum appearing in this determinant is -1 unless $\alpha = \nu$, in which case it is $N\mathfrak{p} - 1$. That is to say,

$$|\Delta|^2 = |\det\{-1 + \delta_{\nu\mu} N\mathfrak{p} \mid \nu, \mu = 1, 2, \cdots, N\mathfrak{p} - 1\}|,$$

where $\delta_{\nu\mu}$ is the Kronecker delta. This determinant may be evaluated by well-known methods and gives the desired result.

Further considerations will be restricted to the special Fourier transforms ϕ_s. The values of $\lambda(\alpha)$ are p-th roots of unity, where p is the rational prime which $\mathfrak{p}$ divides. The values of $\phi_s(\lambda)$ are therefore integers in the extension E obtained by the adjunction of a primitive p-th root of unity to F. The problem is the factorization of the integers $\phi_s(\lambda)$ in the field E. Theorem 1 shows that the divisors of $\phi_s(\lambda)$ are necessarily divisors of p.

It will now be assumed that $\mathfrak{p}$ is a prime of first degree relative to R. This condition prevails throughout the remainder of the paper. Let $\mathfrak{p}_q = \sigma_q(\mathfrak{p})$, $q \varepsilon N$. According to Hilbert's theory of the factorization of primes in normal extensions,[5] the primes $\mathfrak{p}_q$ and $\mathfrak{p}_{q'}$ are distinct when $q \neq q'$ in N and as q runs through the elements of N the primes $\mathfrak{p}_q$ run through the divisors of p in F. It is not difficult to see [6] that $\mathfrak{p}_q$ is completely ramified in E, that is, there is a prime $\mathfrak{P}_q$ in E such that $\mathfrak{p}_q = \mathfrak{P}_q^{p-1}$. As q runs through the elements of N, the primes $\mathfrak{P}_q$ run through the divisors of p in E.

According to Theorem 1, the factorization of $\phi_s(\lambda)$ in E may be written $\phi_s(\lambda) = \Pi_q \mathfrak{P}_q^{\nu(s\|q)}$. The values of the exponents $\nu(s\|q)$ are given by the following theorem.

THEOREM 2. *If $\lambda \neq 1$, then $\nu(s\|q) = (p-1)\{1 - \omega(qs)/n\}$ if $s \not\equiv 0 \pmod{n}$ and $\nu(s\|q) = 0$ if $s \equiv 0 \pmod{n}$, where $\omega(m)$ is the least non-negative integer congruent to $m \pmod{n}$.*

Proof. The elements of Φ are represented by the rational integers $0, 1, 2, \cdots, p-1$, hence

$$\phi_s(\lambda) = \sum_{\nu=1}^{p-1} (\nu/\mathfrak{p})^s \lambda(\nu) = \sum_{\nu=1}^{p-1} (\nu/\mathfrak{p})^s \lambda^\nu(1).$$

[5] D. Hilbert, *loc. cit.*, Chap. X.

[6] D. Hilbert, *loc. cit.*, Chap. XXI.

Since $\lambda \not\equiv 1$, $\lambda(1)$ is a primitive p-th root of unity. Let $\pi = \lambda(1) - 1$. Then [6] $\pi = \Pi_q \mathfrak{P}_q$, $q \in N$. Applying σ_q to the congruence [3] $(\nu/\mathfrak{p})^q \equiv \nu^{(p-1)q/n} \pmod{\mathfrak{p}}$ gives $(\nu/\mathfrak{p}) \equiv \nu^{(p-1)q/n} \pmod{\mathfrak{p}_q}$ and

$$\phi_s(\lambda) \equiv \sum_{\nu=1}^{p-1} \nu^{(p-1)q/n} (1+\pi)^\nu \pmod{\mathfrak{P}_q{}^{p-1}}.$$

Since $\nu^{p-1} \equiv 1 \pmod{p}$, one may replace the exponent $(p-1)qs/n$ by $(p-1)\omega(qs)/n$.

Using the binomial theorem,

$$\phi_s(\lambda) \equiv \sum_{\mu=1}^{p-1} \pi^\mu \sum_{\nu=\mu}^{p-1} \binom{\nu}{\mu} \nu^{(p-1)\omega(qs)/n} + \sum_{\nu=1}^{p-1} \nu^{(p-1)\omega(qs)/n} \pmod{\mathfrak{P}_q{}^{p-1}}.$$

If s is divisible by n, then $\phi_s(\lambda) = -1$ and $\nu(s \parallel q) = 0$. It may be assumed that $s \not\equiv 0 \pmod{n}$. In this case the exponent $(p-1)\omega(qs)/n$ is not divisible by $p-1$, which means that the second sum in the congruence for $\phi_s(\lambda)$ is congruent to zero mod p. It may be dropped mod $\mathfrak{P}_q{}^{p-1}$.

Let $\mu \geqq 1$ and $1 \leqq l \leqq p-2$. It follows from $\binom{\nu}{\mu} \nu^l = (\nu^{\mu+l} + \text{terms of lower degree in } \nu)/\mu!$ that

$$\sum_{\nu=\mu}^{p-1} \binom{\nu}{\mu} \nu^l \begin{cases} \equiv 0 \pmod{p}, & \mu \leqq p-l-2 \\ \not\equiv 0 \pmod{p}, & \mu = p-l-1. \end{cases}$$

Applying this to the congruence for $\phi_s(\lambda)$, one concludes that

$$\phi_s(\lambda) \equiv a\pi^{(p-1)\{1-\omega(qs)/n\}} \pmod{\mathfrak{P}_q{}^{p-(p-1)\omega(qs)/n}},$$

where a is a rational integer $\not\equiv 0 \pmod{p}$ and hence $\not\equiv 0 \pmod{\mathfrak{P}_q}$. Theorem 2 now follows from this congruence.

3. The convolutions. The coefficient of $\lambda(\alpha)$ in the product $\phi_s(\lambda)\phi_t(\lambda)$ will be denoted by $g_s{}^t(\alpha)$. Its explicit value is $\Sigma_\beta(\beta/\mathfrak{p})^s(\{\alpha-\beta\}/\mathfrak{p})^t$, which is an integer in F. Owing to the multiplicative nature of the power residue symbol, $g_s{}^t$ is almost the function $(\alpha/\mathfrak{p})^{s+t}$.

Case 1. $\alpha \not\equiv 0$.

$$g_s{}^t(\alpha) = \Sigma_\beta(\alpha\beta/\mathfrak{p})^s(\{\alpha-\alpha\beta\}/\mathfrak{p})^t = (\alpha/\mathfrak{p})^{s+t} g_s{}^t(1).$$

Case 2. $\alpha = 0$.

$$g_s{}^t(0) = (-1/\mathfrak{p})^t \Sigma_\beta(\beta/\mathfrak{p})^{s+t} = \begin{cases} 0 & , s+t \not\equiv 0 \pmod{n} \\ (-1/\mathfrak{p})^t(N\mathfrak{p}-1), & s+t \equiv 0 \pmod{n}. \end{cases}$$

Using these values of $g_s{}^t(\alpha)$ it follows that

$$\phi_s(\lambda)\phi_t(\lambda) = \begin{cases} g_s{}^t(1)\phi_{s+t}(\lambda) & , s+t \not\equiv 0 \pmod n \\ g_s{}^t(1)\phi_{s+t}(\lambda) + (N\mathfrak{p}-1)(-1/\mathfrak{p})^t, & s+t \equiv 0 \pmod n. \end{cases}$$

The factorization of $g_s{}^t(1)$ in F may be written $g_s{}^t(1) = \Pi_q \mathfrak{p}_q{}^{\nu(s|t\|q)}$, as will be demonstrated in Theorem 3, which gives the values of the exponents $\nu(s \mid t \parallel q)$ in most cases.

THEOREM 3. *If neither s nor t is divisible by n, then*

$$\nu(s \mid t \parallel q) = 1 - [q(s+t)/n] + [qs/n] + [qt/n].$$

Proof. Case 1. $s + t \not\equiv 0 \pmod n$.

$$g_s{}^t(1) = \phi_s(\lambda)\phi_t(\lambda)/\phi_{s+t}(\lambda).$$

The results of Theorem 2 and the relations $\mathfrak{p}_q = \mathfrak{P}_q{}^{p-1}$ and

$$\omega(m) = m - [m/n]n$$

give the desired result.

Case 2. $s + t \equiv 0 \pmod n$. Put $\lambda = 1$ into the expression for $\phi_s(\lambda)\phi_t(\lambda)$. Then $g_s{}^t(1) = -(-1/\mathfrak{p})^t = \pm 1$ and $\nu(s \mid t \parallel q) = 0$.

4. The class group relations. The proof of Theorem 4 may now be given. Let $\mathfrak{K}$ be the ideal class which contains p. For this ideal class we may state from Theorem 3 that $\Pi_q \sigma_q(\mathfrak{K})^{\nu(s|t\|q)}$ is the principal class, provided that neither s nor t is divisible by n. Furthermore, the factorization $p = \Pi_q \mathfrak{p}_q$ shows that $\Pi_q \sigma_q(\mathfrak{K})$ is also the principal class. This means (1) that the first statement regarding $\mathfrak{K}$ is true even if s or t or both are divisible by n and (2) that

$$\Pi_q \sigma_q(\mathfrak{K})^{1-\nu(s|t\|q)}$$

is the principal class. The latter statement is Theorem 4 for the case when $\mathfrak{K}$ is the ideal class containing p.

The generalized theorem on arithmetic progressions [7] assures us that every ideal class of F contains not only one but an infinite number of primes satisfying the conditions on $\mathfrak{p}$ in this paper. Hence Theorem 4 is true for every ideal class of F.

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[7] For the most general statement of this theorem see H. Hasse, *ibid.*, vol. 35 (1926), § 8.

A THEOREM ON UNIFORM CAUCHY POINTS.*[1]

By W. L. Gordon and C. W. McArthur.

The purpose of this paper is to prove a theorem which implies the main results of recent papers of B. J. Pettis, [4], A. Alexiewicz [1], and A. M. Ostrowski [3] concerning the Osgood-Kuratowski-Banach theorem and some of its consequences. The terminology of [4] will be followed throughout this paper.

Let X be any topological space, and Φ any cardinal number. A subset E of X is a I_Φ set if it may be represented as the union of Φ nowhere dense sets; otherwise it is a II_Φ set; E is Φ-residual if its complement $X \setminus E$ is a I_Φ set. Let Y be any completely regular topological space, and of all uniformizations of the topology in Y [5], let $\{V_\alpha \mid \alpha \in A\}$ be one having least cardinal number $|A|$. For each element λ of a fixed directed set Λ, let f_λ be a function on X to Y. A point x_0 in X is a *Cauchy point* of $\{f_\lambda\}$ if (0.1) given any $\alpha \in A$, there is some λ_α such that $f_\lambda(x_0) \in V_\alpha(f_{\lambda_\alpha}(x_0))$ for all $\lambda \geq \lambda_\alpha$. A point x_0 is a *uniform Cauchy point* of $\{f_\lambda\}$ if (0.2) for each $\alpha \in A$ there is some $\lambda_\alpha \in \Lambda$, and an open set G containing x_0, such that $f_\lambda(x) \in V_\alpha(f_{\lambda_\alpha}(x))$ for all $\lambda \geq \lambda_\alpha$ and all $x \in G$. The set of Cauchy points of $\{f_\lambda\}$ will be denoted by C and the set of uniform Cauchy points by C_u. The closure and interior of any set E in a topological space will be denoted by E^* and E^0 respectively.

Our main result may now be stated.

Theorem 1. *Let $|\Lambda|$ represent the least of the cardinal numbers of cofinal subsets*[2] *of Λ. If each f_λ is continuous, and if $\Phi \geqq \max(|A|, |\Lambda|)$, then $C \setminus C_u$ is I_Φ.*

For each $\alpha \in A$ there exists ([5], p. 14) a non-negative real function $\sigma_\alpha(p, q)$ on $Y \times Y$ such that (i) $\sigma_\alpha(p, p) = 0$, (ii) $\sigma_\alpha(p, q) = \sigma_\alpha(q, p)$, (iii) if $\sigma_\alpha(p, q) < 1$ then $p \in V_\alpha(q)$ and $q \in V_\alpha(p)$, and (iv) given $\epsilon > 0$

* Received September 13, 1951.

[1] Part of this paper was written under contract N7-onr-434, Task Order III, Navy Department (the Office of Naval Research).

[2] Equivalently, let $|\Lambda|$ be the least of the cardinal numbers of domains of subnets of the net $\{f_\lambda\}$ in the sense of J. L. Kelley, "Convergence in topology," *Duke Mathematical Journal*, vol. 17 (1950), No. 3.

there is some $\beta \in A$ such that $|\sigma_\alpha(r,s) - \sigma_\alpha(p,q)| < \epsilon$ whenever $r \in V_\beta(p)$ and $s \in V_\beta(q)$. Let $M = [\mu]$ be a cofinal subset of Λ having cardinal number $|\Lambda|$, and for each $\alpha \in A$ and $\mu \in M$, let

$$Q_{\alpha\mu} = X[x: |\sigma_\alpha(f_\lambda(x), f_\mu(x))| \leq \tfrac{1}{2} \text{ for all } \lambda \geq \mu].$$

In [4] it is shown without using the continuity of the f_λ's that $C = \bigcap_\alpha \bigcup_\mu Q_{\alpha\mu}$, and $C_u = \bigcap_\alpha \bigcup_\mu Q^0{}_{\alpha\mu}$. From this it follows that

$$C \setminus C_u \subset E = \bigcup_\alpha \bigcup_\mu (Q_{\alpha\mu} \setminus Q^0{}_{\alpha\mu}).$$

By hypothesis, each f_λ is continuous; hence, by (iv) above, for fixed λ and μ, $|\sigma_\alpha(f_\lambda(x), f_\mu(x))|$ is a continuous function of x. Therefore each $Q_{\alpha\mu}$ is closed, and $Q_{\alpha\mu} \setminus Q^0{}_{\alpha\mu}$ is nowhere dense. Hence E is the union of $|A| \cdot |\Lambda|$ nowhere dense sets. If $|A| \cdot |\Lambda|$ is finite, E is nowhere dense and so obviously I_Φ. If $|A| \cdot |\Lambda|$ is infinite, then $|A| \cdot |\Lambda| = \max(|A|, |\Lambda|) \leq \Phi$, and E is necessarily a I_Φ set. As a subset of a I_Φ set, $C \setminus C_u$ is also I_Φ.

With the same hypotheses on the spaces X and Y as in definitions (0.1) and (0.2), and with $\{f_\lambda\}$ a directed family of functions on X to Y, a *point of almost equi-continuity* of $\{f_\lambda\}$ is a point x_0 satisfying the condition that (0.3), for each $\alpha \in A$ there exists a λ_α and an open set G containing x_0, such that $f_\lambda(x) \in V_\alpha(f_\lambda(x_0))$ for all $\lambda \geq \lambda_\alpha$ and all $x \in G$. The set of all such points will be denoted by C_a. We recall the following result ([4]).

LEMMA 1. *The relations $C_u \supset C^* \cap C_a = C \cap C_a$ always hold, and if f_λ is continuous for each λ in a cofinal subset of Λ, then $C_u = C^* \cap C_a = C \cap C_a$.*

The following corollary is a combined statement of Theorems 1 and 2 of Pettis ([4]), the latter part being an extension of a theorem of Alexiewicz ([1]).

COROLLARY 1.1. *If each f_λ is continuous, and $\Phi \geq \max(|A|, |\Lambda|)$, then $C_u \subset C_a$, and*

(1) *if C is Φ-residual, then C_u and C_a are Φ-residual;*

(2) *if C is a II_Φ set, then $C \cap C_a$ $(= C_u)$ is a II_Φ set.*

By Theorem 1, $C \setminus C_u$ is I_Φ, and since $X \setminus C_u = (C \setminus C_u) \cup (X \setminus C)$ is the union of two I_Φ sets, $X \setminus C_u$ is I_Φ; that is, C_u is Φ-residual. By Lemma 1, $C_u \subset C_a$, so C_a is also Φ-residual. The second part of the conclusion is evident since $C = (C \setminus C_u) \cup C_u$, and $C \setminus C_u$ is I_Φ.

Given any function f_0 on X to Y, let C_0 be the set of *points of continuity*

of f_0; and given f_0 and $\{f_\lambda\}$, let $C(f_0)$ and $C_u(f_0)$ represent the sets of those points for which (0.1) and (0.2) respectively hold when f_{λ_α} is replaced by f_0.

We recall the following lemma ([4]).

LEMMA 2. *For any* f_0 *and any* $\{f_\lambda\}$, *we have*

(1) $C \supset C(f_0)$; (2) $C_0 \supset C(f_0)^0 \cap C_a$;

(3) $C(f_0) \cap C_u \supset C_u(f_0) \supset C(f_0)^0 \cap C_u$;

(4) *if* f_λ *is continuous for each* λ *in a cofinal set, then* $C_0 \supset C_u(f_0)$.

The Osgood theorem now takes the following form.

THEOREM 2. *Given* $\{f_\lambda\}$ *and* f_0, *suppose that each* f_λ *is continuous and that* $\Phi \geq \max\{|A|, |\Lambda|\}$. *Then* $C_u(f_0) \subset C_a \cap C_0$ *and* $C(f_0)^0 \setminus C_u(f_0)$ *is* I_Φ.

From (1) and (3) of Lemma 2 we have $C_u(f_0) \subset C \cap C_u$, and from Lemma 1, $C \cap C_u = C \cap C_a$; thus $C_u(f_0) \subset C_a$. By (4) of Lemma 2, $C_u(f_0) \subset C_0$, hence we clearly have $C_u(f_0) \subset C_a \cap C_0$, or equivalently $C_u(f_0) = C_u(f_0) \cap C_a \cap C_0$. Using 3 of Lemma 2 we have

$$C(f_0)^0 \setminus C_u(f_0) \subset C(f_0)^0 \setminus (C(f_0)^0 \cap C_u) = C(f_0)^0 \setminus C_u \subset C \setminus C_u.$$

Hence by Theorem 1, $C(f_0)^0 \setminus C_u(f_0)$ is I_Φ.

COROLLARY 2.1. *Assuming the hypothesis of Theorem* 2, *then*

(i) *if* $C(f_0)^0$ *is* Φ*-residual then so is* $C_u(f_0) \cap C_a \cap C_0 \cap C(f_0)^0$;

(ii) *if* $C(f_0)^0$ *is* II_Φ *so is* $C_u(f_0) \cap C_a \cap C_0 \cap C(f_0)^0$.

From Theorem 2, $C_u(f_0) = C_u(f_0) \cap C_a \cap C_0$, and since

$$(X \setminus C(f_0)^0) \cup (C(f_0)^0 \setminus C_u(f_0)) = X \setminus (C(f_0)^0 \cap C_u(f_0)),$$

where $X \setminus C(f_0)^0$ and $C(f_0)^0 \setminus C_u(f_0)$ are I_Φ sets, (i) follows. To establish (ii) note that $C(f_0)^0 = (C(f_0)^0 \setminus C_u(f_0)) \cup (C_u(f_0) \cap C(f_0)^0)$, where $C(f_0)^0$ is II_Φ and $C(f_0)^0 \setminus C_u(f_0)$ is I_Φ. Therefore $C_u(f_0) \cap C(f_0)^0$ is II_Φ.

Observe that in the special case of Corollary 2.1 obtained by requiring $C(f_0) = X$, one may conclude in addition that $C_a = C_u = C_u(f_0) \subset C_0$, because in this case parts (3) and (4) of Lemma 2 imply $C_u = C_u(f_0) \subset C_0$ and Lemma 1 implies that $C_u = C_a$.

The set Λ is called *essentially denumerable* if $|\Lambda| \leq \aleph_0$ (e. g. positive integers). The following theorem summarizes the foregoing results in case Λ is essentially denumerable and Y is pseudo-metric. Parts (2), (3), and a consequence of (5) have been stated by Pettis [4] and in less generality by Alexiewicz [1].

THEOREM 3. *Let Y be pseudo-metric and Λ essentially denumerable, and suppose f_λ is continuous for each $\lambda \varepsilon \Lambda$. Then*

(1) $C \setminus C_u$ *is first category;*

(2) *if C is residual so are C_a and C_u;*

(3) *if C is second category so are C_a and C_u.*

If also f_0 is on X to Y, then

(4) $C(f_0)^0 \setminus C_u(f_0)$ *is first category;*

(5) *if $C(f_0)^0$ is residual so is $C_u(f_0) \cap C_a \cap C_0 \cap C(f_0)^0$;*

(6) *if $C(f_0)^0$ is second category so is $C_u(f_0) \cap C_a \cap C_0 \cap C(f_0)^0$.*

If X is second category then so are C_u and C_a in cases (2), (3), *and in the cases* (5), (6) *so is $C_u(f_0) \cap C_a \cap C_0 \cap C(f_0)^0$.*

Following Bourbaki [2] we call X a *Baire Space* if every non-null open set of X is second category (e. g. a locally compact Hausdorff space or a complete pseudo-metric space). In such a case it is clear that any residual set is second category and everywhere dense. Hence

COROLLARY 3.1. *Let Y be a pseudo-metric space, X a Baire space, Λ essentially denumerable, and suppose f_λ is continuous for each $\lambda \varepsilon \Lambda$ and f_0 is on X to Y. If $C(f_0)^0$ is residual, then $C_u(f_0) \cap C_a \cap C_0 \cap C(f_0)^0$ is residual, second category, and everywhere dense.*

In a recent paper A. M. Ostrowski [3] states the following theorem: Let $f(t, x)$ be a continuous function of the point (t, x) for $a < x < b$ and $t \geq T$, and suppose we have $\lim_{t \to \infty} f(t, x) = f(x)$, where $f(x)$ is also continuous in (a, b); then for any $\epsilon > 0$ there exists a subinterval J of (a, b) and a number T_0 such that we have $|f(t, x) - f(x)| < \epsilon$ for $t \geq T_0$ and $x \varepsilon J$.

Using Corollary 3.1, Ostrowski's theorem may be strengthened to the following: Suppose for each $t \geq T$ that $f(t, x)$ is a continuous function on $X =$ real interval (a, b) into a pseudo-metric space Y, and that f is a function on X to Y such that $\lim_{t \to \infty} f(t, x) = f(x)$ for each $x \varepsilon X$. Then $C_a \cap C_u(f) \cap C_0$ is residual, second category, and everywhere dense, and $C_a = C_u(f) \subset C_0$.

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MATRICES WITH INTEGER ELEMENTS IN COMBINATORIAL INVESTIGATIONS.*

By H. J. RYSER.

1. Introduction. Let it be required to arrange n elements into n sets, such that each set contains exactly k distinct elements and such that every pair of sets has exactly $\lambda = k(k-1)/(n-1)$ elements in common $(0 < \lambda < k < n)$. The significance of this n, k, λ combinatorial problem has been described in detail in [2]. The problem for $n = N^2 + N + 1$, $k = N + 1$, $N \geqq 2$, and $\lambda = 1$ is equivalent to finding a projective plane with $N + 1$ points on a line. For $n = 4N - 1$, $k = 2N - 1$, and $\lambda = N - 1$, the problem reduces to a determination of an Hadamard matrix of order $4N$.

Let $x_1, \cdots, x_n$ denote the n elements and let $s_1, \cdots, s_n$ denote the n sets formed from these elements. The elements $x_1, \cdots, x_n$ may be listed in a row and the sets $s_1, \cdots, s_n$ in a column. One forms the incidence matrix A in the usual way by inserting 1 in row i and column j if the element x_j belongs to the set s_i and 0 otherwise. The matrix A of order n is composed entirely of zeros and ones and satisfies the matric equation $AA' = B$. Here A' denotes the transpose of A, and the matrix B has k in the main diagonal and λ in all other positions. Moreover, for the incidence matrix A, one readily proves that $A'A = B$. Clearly, the matrix B is congruent to the identity, relative to the field of rational numbers. The utilization of this fact has resulted in a proof of the impossibility of the n, k, λ problem for large classes of n, k, and λ ([1], [2]).

Consider now two n by n symmetric matrices A and B, with integer elements. The matrix B is *integrally represented by* A provided there exists a matrix C with integer elements such that $C'AC = B$. In particular, B is integrally represented by the identity provided there exists an integral A such that $AA' = B$.

Let B now denote the n by n symmetric matrix with k in the main diagonal and $\lambda = k(k-1)/(n-1)$ in all other positions. If the n, k, λ problem has a solution, then clearly B must be integrally represented by the identity. The purpose of the present paper is to show that for certain n, k, and λ, the converse proposition is also valid.

* Received September 20, 1951.

Unfortunately, the classical problem of deciding whether or not one matrix is integrally represented by another is not entirely solved. An excellent account of this and related arithmetic topics may be found in [3].

2. Theorems concerning the n, k, λ problem. The following is a generalization of Theorem 3 derived in [1].

THEOREM 2.1. *Let B be the symmetric matrix of order n, with k in the main diagonal and $\lambda = k(k-1)/(n-1)$ in all other positions $(0 < \lambda < k < n)$. Suppose that $AA' = A'A = B$, where A has integral elements. Then A is composed entirely of zeros and ones, or entirely of zeros and minus ones. Consequently the matrix A yields a solution of the n, k, λ problem.*

Let s_i denote the sum of column i of the matrix $A = [a_{rs}]$. Then

$$\sum_{i=1}^{n} a_{ji} s_i = k^2, \tag{1}$$

where $j = 1, \cdots, n$. Adding, one obtains

$$\sum s_i^2 = nk^2. \tag{2}$$

It is clear from $A'A = B$ that $|s_j| = |\sum_i a_{ij}| \leqq \sum_i a_{ij}^2 = k$, whence $s_j^2 \leqq k^2$. But then (2) implies $|s_j| = k$. But if $s_j = k$, then all of the a's in the j-th column of A consist of zeros and ones. On the other hand, if $s_j = -k$, then all of the a's in the j-th column are zeros and minus ones. Thus the columns of A are either composed of zeros and ones, or zeros and minus ones. Since, however, the inner product of two column vectors of A is the positive integer λ, the matrix does not contain columns of both varieties.

Actually, the restriction of the normality of A in Theorem 2.1 is unnecessarily severe. However, any restriction imposed on the transforming matrix is undesirable. This difficulty is overcome subsequently for certain values of n, k, and λ.

THEOREM 2.2. *Let B denote the symmetric matrix of order n, with k in the main diagonal and $\lambda = k(k-1)/(n-1)$ in all other positions $(0 < \lambda < k < n)$. Suppose further that k and $k-\lambda$ are relatively prime, that $k-\lambda$ is odd, and that $AA' = B$, where A is integral. Then upon suitable multiplication of the columns of A by ± 1, the matrix A is transformed into an incidence matrix composed entirely of zeros and ones. Consequently A yields a solution of the n, k, λ problem.*

Again let s_i denote the sum of column i of A, and let t_{ij} denote the inner product of the i-th and j-th column vectors of A. We first derive a basic relationship connecting the quantities t_{ij} and the products $s_i s_j$, namely

$$k^2 t_{ij} = \lambda s_i s_j + k^2(k-\lambda)\delta_{ij}, \tag{3}$$

where δ_{ij} is the Kronecker delta.

The matrix A is nonsingular, with determinant $|A| = \pm k(k-\lambda)^{(n-1)/2}$. For if column one of B is subtracted from each of the other columns, and if to row one there is then added each of the other rows, it readily follows that $|B| = k^2(k-\lambda)^{n-1}$. Thus $AA' = B$ implies that

$$A'A = A^{-1}BA. \tag{4}$$

Let A_{rs} denote the cofactor of the element a_{rs} of the matrix A. Then comparing elements in the (u, v) positions on both sides of (4), one obtains

$$|A| t_{uv} = \lambda \sum_{i,j} A_{iu}a_{iv} + (k-\lambda) \sum_{i} A_{iu}a_{iv}.$$

But $\sum_i A_{iu}a_{iv} = |A| \delta_{uv}$, and equation (1) implies that

$$|A| s_u = k^2 \sum_i A_{iu}. \tag{5}$$

Hence (3) follows.

Any column of A may be multiplied by -1 and the equation $AA' = B$ remains valid. Hence we may suppose without loss of generality that $s_i \geqq 0$. Now $|A| = \pm k(k-\lambda)^{(n-1)/2}$, and by hypothesis, k and $k-\lambda$ are relatively prime. Hence by equation (5), it follows that each $s_i \equiv 0 \bmod k$. Thus we may write $s_i = u_i k$. Moreover, from (3) it follows that

$$t_{ii} = \lambda u_i^2 + (k-\lambda). \tag{6}$$

Let us now suppose that some $u_i = 0$. Then $s_i = 0$, and by (6) it follows that $t_{ii} = k - \lambda$. But

$$0 \equiv s_i^2 \equiv a_{1i}^2 + \cdots + a_{ni}^2 \equiv k - \lambda \pmod 2,$$

and this contradicts the assertion that $k-\lambda$ is odd. Hence each $u_i \neq 0$. But $s_1^2 + \cdots + s_n^2 = k^2 n$, whence $u_1^2 + \cdots + u_n^2 = n$. Hence it follows that each $u_i = 1$ and thus $s_i = k$ for all i. But then $t_{ii} = k$, and hence each a_{ij} must equal 0 or 1. Thus A is an incidence matrix.

The preceding theorem is of interest for the special case of projective planes and Hadamard matrices.

THEOREM 2.3. *Let B denote the symmetric matrix of order $n = N^2 + N + 1$, with $N+1$ in the main diagonal and 1 in all other positions. Let N be an odd integer > 1. Then the assertion that there exists a projective plane with $N+1$ points on a line is equivalent to the assertion that the matrix B is integrally represented by the identity.*

THEOREM 2.4. *Let B denote the symmetric matrix of order $n = 4N - 1$, with $2N-1$ in the main diagonal and $N-1$ in all other positions. The assertion that Hadamard matrices of all orders congruent to zero modulo 4 exist is equivalent to the assertion that the matrix B is integrally represented by the identity, for every odd integer N.*

If B is integrally represented by the identity for every odd integer N, then by Theorem 2.2, there must exist Hadamard matrices of order $4N$. To construct the Hadamard matrix of arbitrary order congruent to zero modulo 4, it is only necessary to form the direct product with the Hadamard matrix whose order is a suitable power of 2 (*cf.* [4]).

Theorem 2.2 is not necessarily valid for $k - \lambda$ an even integer. In this case certain of the u_i may actually equal zero. If $u_1, u_2, \cdots, u_t$ constitute those u_i equal to zero, then equation (3) merely implies that $A'A$ is a direct sum of two matrices, one factor of which is $(k-\lambda)I$, where I is the identity matrix of order t.

To furnish an example, let H be an Hadamard matrix of order $N = 2$ or $N \equiv 0 \pmod 4$. The first column of H is to be composed entirely of plus ones. Consider now the matrix of order $N^2 + N$, which is the direct sum of the matrix H, taken $N+1$ times. Let this matrix be bordered by a row vector

$$(1, 0, \cdots, 0, 1, 0, \cdots, 0, \cdots, 1, 0, \cdots, 0)$$

of $N^2 + N$ components, where each of the sequences of zeros contains $N-1$ members. Let the resulting array be bordered by a column vector of $N^2 + N + 1$ components, consisting of 0 in the first position and 1 in all other positions. The resulting matrix A is of order $N^2 + N + 1$, and satisfies the matric equation $AA' = B$. However, the matrix A is not the incidence matrix of a finite projective plane.*

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A GENERAL THEORY OF RADICALS.*

I. Radicals in Complete Lattices.

By S. A. AMITSUR.

The various radicals which have been hitherto defined by various authors (Artin, Levitzki, Jacobson, Brown-McCoy, etc.) constitute an important tool in the study of the structure of rings. The theory of radicals was recently extended to non associative and non distributive rings as well as to more general structures. The similarities which exist between some of these radicals (in the underlying definitions and reasoning) have been already observed by B. Brown and N. H. McCoy, and they developed in [4] a theory for radicals in groups which, in particular cases, yields some of the known radicals in the theory of rings, but bears no relations to others. The purpose of the present paper is to give an axiomatic study of radicals. In order to achieve the greatest possible generality it was found suitable to develop the theory of radicals for complete lattices.[1] The axiomatic approach and the general results obtained here will be applied in a subsequent paper where also the the results of [4] will be incorporated in our general theory.

One readily observes that each of the radicals which have been hitherto defined in rings is connected with some ring-property which is invariant under ring-homomorphism.[2] Thus the Jacobson-Perlis radical ([3]) grows up out of the property of quasi-regularity. Generally, by a π-radical N of a ring S is meant a maximal ideal N of S possessing a given property π of this type, and such that the quotient S/N is free of non zero ideals with the same property. It turns out that the theory of radicals is based on the following simple consequence of the homomorphism-invariance of the property π: If A, B, C, are any three ideals in a ring S such that the quotient ring A/B has the property π then $(A, C)/(B, C)$ has the same property.[3] This fact, and the method

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[1] I am indebted to Prof. R. Baer for a remark which led to the present general treatment of the theory of radicals.

[2] Compare with [4], Theorem 2.

[3] This follows immediately since A/B is homomorphic with $A/A \cap (B, C) \cong (A. B, C)/(B, C) \cong (A, C)/(B, C)$.

which has been applied by Baer for defining his Lower Radical in [1] are the starting points of the theory of radicals developed in this paper.

It is found convenient to formulate our conditions and results not in terms of properties but rather in terms of binary relations ρ between elements of a complete lattice. Thus in the case of ideals in a ring S and a ring-property π we write $A \rho B$ if $A \supseteq B$ and the quotient ring A/B has the desired property π. In particular, a ρ-radical r of a lattice L is an element $r \in L$ such that $r \rho 0$ and $x \rho r$ implies $x = r$. In this notation the above mentioned condition may be formulated as follows: if $a \rho b$ then $(a \cup c) \rho (b \cup c)$.

In Section **1** the definition of Baer's lower radical is extended to complete lattices with binary relations of the preceding type. The discussion of the existence of the radical is carried out in Section **2**. In order to cover the problem of the connection between the radical of a ring and the radical of its ideals in the theory of radicals of associative rings the whole theory is extended in Section **4** to more general structures to be called complete pseudo-lattices. Some related questions and the dual development of the present theory is dealt with in Sections **5** and **6**. The last section deals with an application to lattices in which multiplication is defined. We obtain an extension of Baer's lower radical of [1] to such lattices. The dual definition of this radical yields the maximal idempotent element of such lattices. The latter is an extension of the idempotent kernel of rings and semi-groups defined by J. Levitzki in [2].

1. The upper radical.

Notations. Let M be a complete lattice. We denote by I_M and by 0_M the unit and the zero of M. When no confusions are expected, the subscript M will be omitted. Lattices and sublattices will always mean complete lattices and complete sublattices. By an M-interval $[a, b]$ is meant the set $\{x; x \in M, a \leq x \leq b\}$. The notation $\mathrm{Sup}[x; \cdots]$ and $\mathrm{Inf}[x; \cdots]$ will be used to denote the greatest lower bound and respectively the least upper bound of the elements x subjected to a condition which will replace the dots in the brackets. We refer to the relation $\geq$ of M as to an inclusion relation, and we say that a includes b if $a \geq b$.

We consider a set of sublattices $\{L\}$ of M such that if $a, b \in L$, where $L \in \{L\}$, the L-interval $[a, b]$ belongs also to the set $\{L\}$.

Definition 1. A binary relation ρ defined in M is called an *H-relation* in M if ρ satisfies:

(A) If $a \rho b$; $a, b \in M$, then $a \geq b$.

(B) $a \rho a$ for every $a \in M$.

(C) If $a \rho b$ and $c \geq b$ then $(a \cup c) \rho c$.

Remark. Let $a \rho b$. Since $b \cup c \geq b$, it follows by (C) that $(a \cup c) \rho ((b \cup c)$ for every $c \in M$.

When $a \rho b$ and $a, b \in L$ we write $a \rho b$ *in* L, and a is said to be a ρ-*element over* b *in* L. An element $a \in L$ is said to be a ρ-*element in* L if $a \rho 0_L$.

A sublattice L of M is said to be ρ-*semi simple* in M if L does not possess non zero ρ-elements, i. e. $x \rho 0_L$ in L holds only for $x = 0_L$. Let $a \leq b$ be two elements of L. If the L-interval $[a, b]$ is ρ-semi simple in M, we write $a \bar{\rho} b$ in L. The element b is said to be a $\bar{\rho}$-*element over* a *in* L. If $a \bar{\rho} I_L$, a is said to be a $\bar{\rho}$-*element in* L.

Definition 2. An element $r \in L$ is called a ρ-*radical* in L if r is both a ρ-element in L and a $\bar{\rho}$-element in L.

Example. Let M be the lattice of the ideals of an associative ring S. The relation ν in M defined to be: $a \nu b$ if $a^n \leq b \leq a$ for some integer n is readily seen to be an H-relation in M. In this example, the $\bar{\nu}$-elements are the radical-ideals defined by Baer in [1] and the ν-radical in M is the nil-potent radical of the ring S (in case it exists!).

Unless otherwise stated, binary relations to be considered hereafter will be H-relations and the sublattices of M will be restricted to the set $\{L\}$ considered above.

Define inductively the following chain of elements in L: $u_0(L, \rho) = 0_L$, $u_1(L, \rho) = \mathrm{Sup}[p; p \rho 0_L \text{ in } L]$, $u_\lambda(L, \rho) = \mathrm{Sup}[u_\nu(L, \rho);\ \nu < \lambda]$ for limit ordinal λ, and $u_\lambda(L, \rho) = \mathrm{Sup}[p; p \rho u_{\lambda-1}(L, \rho) \text{ in } L]$ for non limit ordinal λ. Thus the chain $\{u_\lambda\}$ [4] is a well-defined non-decreasing chain of elements of L.

LEMMA 1.1. *There is an ordinal* τ *such that* $u_\tau(L, \rho) = u_\sigma(L, \rho)$ *for every ordinal* $\sigma \geq \tau$ *and* $u_\nu(L, \rho) < u_\mu(L, \rho)$ *for* $\nu < \mu \leq \tau$ *(if* $\tau > 1$*).*

Since the chain $\{u_\lambda\}$ is a subset of L it is readily verified that there exists a minimal τ such that $u_\tau = u_{\tau+1}$. The rest of the lemma follows now immediately by the definition of the chain $\{u_\lambda\}$ and by the minimality of τ.

The element $u_\lambda(L, \rho)$ is called the λ-*th* ρ-*radical* of L, and u_τ, the element of the preceding lemma, is called the *upper* ρ-*radical* of L (in M). This element will be denoted by $u(L, \rho)$.[4]

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[4] When no confusion is expected u_λ, will replace $u_\lambda(L, \rho)$. Similarly u and r will replace $u(L, \rho)$ and $r(L, \rho)$.

The object of the present section is to determine the relation between the upper ρ-radical of L and the $\bar{\rho}$-elements of L.

The following fact will be often used: if $a \,\bar{\rho}\, b$ in L, then $a \,\bar{\rho}\, b$ in every sublattice of L which contains both a and b.

The following lemma is fundamental:

LEMMA 1.2. *Let p, q be two elements of the L-interval $[a, b]$ such that $q \,\bar{\rho}\, b$ and $p \,\rho\, a$; then $q \geq p$.*

Proof. It follows by (C) that $(p \cup q) \rho\, q$ in L. Since $q \,\bar{\rho}\, b$ and $b \geq p \cup q \geq q$, $p \cup q = q$. That is $q \geq p$.

In particular if $a = I_L$ and $b = 0_L$, we have

COROLLARY 1.1. *Each $\bar{\rho}$-element in L includes every ρ-element of L.*

A simple consequence of the preceding corollary is:

THEOREM 1.1. *Every sublattice L possesses at most one ρ-radical.*

Indeed, if r_1 and r_2 are two ρ-radicals of L, then since $r_1 \,\bar{\rho}\, I$ and $r_2 \,\rho\, 0$, $r_1 \geq r_2$. Similarly $r_2 \geq r_1$. Thus $r_1 = r_2$.

If L possesses a ρ-radical, in the light of the preceding theorem we refer to this radical as *the ρ-radical* of L and denote it by $r(L, \rho)$.

Since $r(L, \rho)$ is a $\bar{\rho}$-element it follows by Corollary 1.1 that

COROLLARY 1.2. *If $r(L, \rho)$ exists in L, then $r(L, \rho)$ is the maximal ρ-element of L.*

LEMMA 1.3. *Let $Q = \{q\}$ be a set of $\bar{\rho}$-elements in L, then $t = \mathrm{Inf}[q; q \in Q]$ is also a $\bar{\rho}$-element in L.*

Let $p \in L$ be such that $p \,\rho\, t$. Since $q \geq t$, $q \in Q$, it follows by Lemma 1.2 that $q \geq p$. This holds for every $q \in Q$; hence $t \geq p$. This proves that t is a $\bar{\rho}$-element.

The set of all $\bar{\rho}$-elements of L is non vacuous, since evidently I_L is a $\bar{\rho}$-element in L. We obtain, therefore, by the preceding lemma,

COROLLARY 1.3. *The meet of all $\bar{\rho}$ elements of L is the minimal $\bar{\rho}$-element in L.*

The following is the main theorem of the present section.

THEOREM 1.2. *The upper ρ-radical $u(L, \rho)$ is the minimal $\bar{\rho}$-element of L.*

Proof. Since by Lemma 1.1 $u(L,\rho) = u_\tau = u_{\tau+1}$, it follows by definition of $u_{\tau+1}$ that $u(L,\rho)$ is a $\bar{\rho}$-element in L. Hence, if $m = \mathrm{Inf}[q; q$ is a $\bar{\rho}$-element in $L]$, $m \leq u(L,\rho)$. To prove that $u(L,\rho) \leq m$, it is sufficient to show in the light of Lemma 1.1 that $m \geq u_\nu(L,\rho)$ for every ordinal ν. By the preceding corollary m is a $\bar{\rho}$-element in L, hence by Corollary 1.1 $m \geq p$ for every ρ-element p in L. This yields $m \geq u_1(L,\rho)$. Let $m \geq u_\nu(L,\rho)$ for every ordinal $\nu < \lambda$. For limit ordinal λ it is evident that $m \geq u_\lambda$. If λ is not a limit ordinal, since $m \geq u_{\lambda-1}$ it follows by Lemma 1.2 that $m \geq p$ for every p which is a ρ-element over $u_{\lambda-1}$. This implies $m \geq u_\lambda$, and the proof is completed.

THEOREM 1.3. *A necessary and sufficient condition that $r(L,\rho)$ exist is that $u(L,\rho)$ is a ρ-element, and in this case $r(L,\rho) = u(L,\rho)$.*

Proof. In view of the preceding theorem, $u(L,\rho)$ is a $\bar{\rho}$-element. Hence, if $u(L,\rho)$ is a ρ-element, $u(L,\rho)$ is the ρ-radical of L. Conversely, let $r(L,\rho)$ exist in L. Since $r(L,\rho)\bar{\rho}\, I$ it follows by Theorem 1.2 that $r(L,\rho) \geq u(L,\rho)$. On the other hand, since $r(L,\rho)$ is a ρ-element, $u(L,\rho) \geq u_1(L,\rho) \geq r(L,\rho)$. Thus $r(L,\rho) = u(L,\rho)$, and the latter is therefore a ρ-element in L.

Since $r(L,\rho) = u_1(L,\rho)$ we have

COROLLARY 1.4. *If $r(L,\rho)$ exists, it is the maximal ρ-element of L.*

A relation between the upper ρ-radicals of two lattices is given in the following theorem:

THEOREM 1.4. *If $L \supseteq L'$ such that $0 = 0'$,[5] then $u(L,\rho) \geq u(L',p)$.*

The proof is achieved by showing inductively that $u(L,\rho) \geq u_\lambda(L',\rho)$. If $p\,\rho\,0$ in L', the same relation holds in L; hence $u(L,\rho) \geq u_1(L',\rho)$. For a limit ordinal λ it is evident that $u(L,\rho) \geq u_\lambda(L',\rho)$ if $u(L,\rho) \geq u_\nu(L',\rho)$ holds for every $\nu < \lambda$. Let λ be not a limit ordinal. If $p\,\rho\,u_{\lambda-1}(L',\rho)$, since p and $u_{\lambda-1}(L',\rho)$ belong to L, this relation holds also in L. Hence, from $u(L,\rho) \geq u_{\lambda-1}(L',\rho)$ it follows by Lemma 1.2 that $u(L,\rho) \geq p$. This yields that $u(L,\rho) \geq u_\lambda(L',\rho)$. q. e. d.

THEOREM 1.5. *Denote by L_a the L-interval $[a, I]$. If $u(L,\rho) \geq a$, then $u(L,\rho) = u(L_a,\rho)$.*

Since $u(L,\rho)\bar{\rho}\, I$ in L, the same holds in the interval L_a since $u(L,\rho) \geq a$.

[5] 0 and 0′ denote respectively the zeros of L and L'.

Hence, Theorem 1.2 yields $u(L,\rho)\geq u(L_a,\rho)$. The same reasoning implies that $u(L_a,\rho)\bar{\rho}\, I$ in L, since this holds in L_a; hence $u(L_a,\rho)\geq u(L,\rho)$ and the theorem is proved.

A relation between the upper radicals of two different relations ρ_1 and ρ_2 is discussed in

THEOREM 1.6. *If either* 1) $a\,\rho_1\, b$ *implies* $a\,\rho_2\, b$, *or* 2) $b\,\bar{\rho}_2\, I$ *in* L *implies* $b\,\bar{\rho}_1\, I$ *in* L, *holds then* $u(L,\rho_1)\leq u(L,\rho_2)$.

We first show that (1) implies (2). Indeed, if $x\,\bar{\rho}_2\, I$ holds and I is not a $\bar{\rho}_1$-element over x, then there is a $p\,\varepsilon\, L$ such that $p>x$ and $p\,\rho_1\, x$. Hence (1) implies $p\,\rho_2\, x$ which is a contradiction. Now assume that (2) holds. Since $u(L,\rho_2)\bar{\rho}_2\, I$ it follows that $u(L,\rho_2)\bar{\rho}_1\, I$; hence by Theorem 1.2 $u(L,\rho_1)\leq u(L,\rho_2)$.

Hence,

COROLLARY 1.5. *If* $a\,\rho_1\, b$ *implies* $a\,\rho_2\, b$, *and* $b\,\bar{\rho}_1\, I$ *implies* $b\,\bar{\rho}_2\, I$, *then* $u(L,\rho_1)=u(L,\rho_2)$.

2. The ρ-radical. This section deals with the existence of the ρ-radical $r(L,\rho)$. For $a\,\varepsilon\, L$, we denote by Q_a the L-interval $[0,a]$. If $u(Q_a,\rho)<a$, then $[u(Q_a,\rho),a]$ is a non zero ρ-semi simple L-interval. If $[x,a]$ is a non-zero ρ-semi simple L-interval in L, then $u(Q_a,\rho)\leq x<a$. Hence

LEMMA 2.1. $u(Q_a,\rho)=a$ *if and only if none of the non zero* L*-intervals* $[x,a]$, $x<a$, *is* ρ*-semi simple.*

Since $L\supseteq Q_a$ and both have the same zero, it follows by Theorem 1.4:

LEMMA 2.2. $u(L,\rho)\geq u(Q_a,\rho)$.

LEMMA 2.3. *If* $a\,\bar{\rho}\, b$ *in* L, *and* $b\,\bar{\rho}\, c$ *in* L, *then* $a\,\bar{\rho}\, c$ *in* L.

Indeed, if $c\geq p\geq a$ such that $p\,\rho\, a$, $(p\cup b)\rho\, b$. Since $c\geq p\cup b\geq b$ and $c\,\bar{\rho}\, b$, it follows that $p\cup b=b$, i. e., $b\geq p\geq a$. But then $a\,\bar{\rho}\, b$ implies $p=a$. This proves that $a\,\bar{\rho}\, c$ in L.

The main theorem of the present section is

THEOREM 2.1. *The* ρ*-radical exists in every lattice* L *of* M *if and only if* ρ *satisfies*

(D) *For every* $a,b\,\varepsilon\, L$ *such that* $a<b$ *and such that* b *is not a* ρ*-element over* a, *there is an element* $c\,\varepsilon\, L$, *such that* $c\,\bar{\rho}\, b$ *in* L.

Proof. Let ρ be an H-relation which satisfies (D). If $u(L,\rho)$ were not a ρ-element, then there is $u(L,\rho) > c \geq 0$ such that $c\,\bar{\rho}\,u(L,\rho)$. By Theorem 1.2 $u(L,\rho)\bar{\rho}\,I$ in L. Hence the preceding lemma yields $c\,\bar{\rho}\,I$. Thus $u(L,\rho) > c$ contradicts Theorem 1.2. This proves that $u(L,\rho)$ is a ρ-element, and therefore we obtain by Theorem 1.3 that $u(L,\rho) = r(L,\rho)$ and the latter exists.

Conversely, let ρ be an H-relation for which the ρ-radical exists in every lattice L (of the set $\{L\}$). Let $b > a$ and b be not a ρ-element over a in L. The L-interval $[a,b]$ belongs to the set $\{L\}$, hence its ρ-radical r exists. Evidently, r is the required element, i. e., $b > r \geq a$ and $r\,\bar{\rho}\,b$ in L.

Remark 2.1. From the first part of the preceding proof it follows that if (D) holds in some lattice L only for the element $a = 0$, then the ρ-radical $r(L,\rho)$ of this lattice L exists.

Definition 3. An H-relation which satisfies (D) is called an *R-relation.*

THEOREM 2.2. *Let ρ be an R-relation. Let $\{a_\alpha\}$ be a set of elements in L which are ρ-elements over b, $b \in L$; then* Sup a_α *is also a ρ-element over b.*

Proof. Let r be the ρ-radical of the L-interval $[b,a]$, where $a = \operatorname{Sup} a_\alpha$. Since $r\,\bar{\rho}\,a$, it follows by Lemma 1.2 that $r \geq a_\alpha$. Hence $r \geq a$. Thus $r = a$, and therefore $a\,\rho\,b$.

One may replace (D) by the conditions of the following theorem.

THEOREM 2.3. *The relation ρ is an R-relation if and only if ρ satisfies the following two conditions:*

(D_1) *If $a\,\rho\,b$ and $b\,\rho\,c$ then $a\,\rho\,c$ (Transitivity).*

(D_2) *If $a_1 \leq a_2 \leq \cdots$ is an ascending well ordered sequence of ρ-elements over b, then* Sup a_i *is also a ρ-element over b.*

Proof. Let ρ satisfy the conditions of the theorem. Applying (D_2) to the ρ-elements of the lattice L (in the case $b = 0_L$), one obtains readily by (D_2) the existence of a maximal ρ-element r. This element r is also a $\bar{\rho}$-element in L, for if $x \in L$, $x > r$ such that $x\,\rho\,r$ then since $r\,\rho\,0$, it follows by (D_1) that $x\,\rho\,0$ which contradicts the maximality of r. This proves that $r = r(L,\rho)$. Hence by Theorem 2.1 it follows that ρ satisfies (D) and, ρ is an R-relation.

Now let ρ be an R-relation. Condition (D_2) is a simple consequence of

Theorem 2. 2. To prove the validity of (D_1), let $r(C)$ be the ρ-radical of the M-interval $[c, a] = C$, where $a \rho b$ and $b \rho c$. Since $b \rho c$, $r(C) \geq b$, and we obtain, therefore, by Theorem 1. 5 that $r(C)$ is also the ρ-radical of the M-interval $[b, a]$. Since $a \rho b$, $r(C) \geq a$. Thus $r(C) = a$, which proves that $a \rho c$. This completes the proof of the theorem.

The reasoning of the first part of the preceding proof yield also the following corollaries.

COROLLARY 2. 1. *If ρ is any relation in M (not necessarily an H-relation) which satisfies* (A), (B), (D_1), *and condition* (D_2) *only in the special case $b = 0_M$, then M possesses maximal ρ-elements, and each maximal ρ-element is a ρ-radical in M.*

In particular, it follows by the preceding proof that if ρ is an H-relation which satisfies (D_1), then every sublattice L either possesses the ρ-radical or does not contain maximal ρ-elements. Hence

COROLLARY 2. 2. *If ρ is an H-relation which satisfies* (D_1), *then every lattice L which satisfies the ascending chain condition for ρ-elements possesses a ρ-radical.*

Example. Let M be the lattice of the ideals of an associative ring S. Consider the following relation between the iedals of S: $a \rho b$ if the quotient ring a/b is quasi regular in the sense of Jacobson ([3]). One readily proves that ρ is an H-relation which satisfies (D_1) and (D_2). Thus the theorems of the present section yield the existence of the Jacobson-Perlis radical.

It is readily seen that the H-relation ν defined in the preceding section satisfies (D_1) and need not satisfy (D_2).

Let L be a fixed sublattice of M. Suppose we distinguish among the elements of L a class $\mathfrak{Z}$, which contains the unity I, of special elements of L. We prove that:

THEOREM 2. 4. *If for every $\bar{\rho}$-element a in L which is not a ρ-element, there exists a special $\bar{\rho}$-element q of $\mathfrak{Z}$ such that $a > a \cap q$ then the ρ-radical of L exists and it is the meet of all the special $\bar{\rho}$-elements of $\mathfrak{Z}$.*

Proof. Let $m = \mathrm{Inf}[q; q \in \mathfrak{Z}, q \text{ is a } \bar{\rho}\text{-element in } L]$. Since $I \in \mathfrak{Z}$, m is well defined. It follows by Lemma 1. 3 that m is a $\bar{\rho}$-element in L. If m is not a ρ-element, then $m > m \cap q$ for some $\bar{\rho}$-element q of $\mathfrak{Z}$. But this contradicts the definition of m. Hence m is the ρ-radical of L. q. e. d.

3. Complete pseudo-lattices. There are still some general aspects in the theory of radicals of rings whose generalization is not covered by the theory developed in the preceding sections. This section deals with one of these problems, the relation between the radical of a ring and the radicals of its ideals. To this end the notion of complete pseudo-lattices is introduced.

Definition 4. A set $\mathfrak{M}$ is called a complete *pseudo lattice* if a binary relation $\geq$ (inclusion) is defined in $\mathfrak{M}$ which satisfies

1) $0 \in \mathfrak{M}$ such that $x \geq 0$ for every $x \in \mathfrak{M}$.

2) Every $\mathfrak{M}$-interval $[a, b] = \{x; x \in \mathfrak{M}, a \leq x \leq b)\}$ is a complete lattice with regard to the inclusion relation.

Note that the relation $\geq$ defined in $\mathfrak{M}$ must be reflexive and antisymmetric but need not be transitive, as can be seen by the following example: Let $\mathfrak{M}$ be the set of all subrings of a ring S. For $a, b \in \mathfrak{M}$, we write $a \geq b$ if b is an *ideal* in the subring a of S. The interval $[b, a]$ of $\mathfrak{M}$ can be identified with the complete lattice of the ideals of the quotient ring a/b.

By an *H-relation* ρ *in* $\mathfrak{M}$ we mean a relation ρ defined in $\mathfrak{M}$ which is an H-relation in every $\mathfrak{M}$-interval and satisfies the condition

(C_1) If $a \cup b$, $a \cap b$ are defined in $\mathfrak{M}$, then $a \bar{\rho} (a \cup b)$ implies $(a \cap b) \bar{\rho}\, b$ in $\mathfrak{M}$.

For complete lattices $\mathfrak{M}$, condition (C_1) is a consequence of (C). Indeed if $(a \cap b) \bar{\rho}\, b$ does not hold in $\mathfrak{M}$, then $p \rho (a \cap b)$ for some $b \geq p > a \cap b$. Hence by (C) $(p \cup a) \rho\, a$, which implies $p \cup a = a$, i. e., $a \geq p$. This together with $b \geq p$ contradicts $p > a \cap b$. This proof does not work in the general case, since $p \cup a$ may not exist in $\mathfrak{M}$.

We denote by $r(n, m)$ the ρ-radical (if it exists) of the $\mathfrak{M}$-interval $[n, m]$. The aim of the present section is to prove

THEOREM 3.1. *If $r(n, m)$ exists, then $r(q, m)$ exists for every $q \in [m, n]$, and $r(q, m) = q \cap r(n, m)$ if and only if ρ satisfies*

(E_1) *If $a \rho\, b$ then $c \rho\, b$ for every $c \in [b, a]$.*[6]

(E_2) *If $a \bar{\rho}\, b$ then $a \bar{\rho}\, c$ for every $c \in [a, b]$.*

Proof. Suppose ρ satisfies the requirements of this theorem. Let $r = r(n, m)$ exist. Since $r \geq q \cap r \geq n$ and $r \rho\, n$, (E_1) yields $(q \cap r) \rho\, n$.

[6] Note that if $c \in [b, a]$, the $\mathfrak{M}$-interval $[b, c]$ is not necessarily a subinterval of $[b, a]$.

Now $m \geq q \cup r \geq r$ and $r \bar{\rho} m$; hence (E_2) yields $r \bar{\rho}(q \cup r)$, which implies by (C_1) that $(q \cap r) \bar{\rho} q$. This proves that $q \cap r(n, m)$ is the ρ-radical $r(q, n)$.

The necessity of (E_1) and (E_2) follows immediately since both conditions are the particular cases $r(b, a) = a$ and $r(a, b) = a$ of the statement of the theorem.

4. The mapping $a \to r(a)$. Let L be a sublattice of M, and let ρ be an R-relation in M. For every $a \in L$ we denote by $r(a)$ the ρ-radical of the L-interval $[a, I]$. The correspondence $a \to r(a)$ is encountered in many problems in the theory of rings, e. g. the correspondence between primary ideals and their prime ideals. A general property of this correspondence is treated in this section.

THEOREM 4. 1. *Let ρ be an R-relation satisfying (E_1) and the condition (F) $(a \cup b) \rho b$ implies $a \rho (a \cap b)$. Then the correspondence $a \to r(a)$ is an idempotent meet homomorphism of L onto itself.*

Proof. The mapping $a \to r(a)$ is isotone. For let $a \geq b$. Since $r(a) \bar{\rho} I$ in the L-interval $[a, I]$, the same holds in the L-interval $[b, I]$. Thus Theorem 1. 2 yields $r(a) \geq r(b)$.

Now let $a_1, a_2 \in L$. Put $r_1 = r(a_1)$ and $r_2 = r(a_2)$. Since $a_i \geq a_1 \cap a_2$, $r_i \geq r(a_1 \cap a_2)$, and therefore $r_1 \cap r_2 \geq r(a_1 \cap a_2)$. Since $r_1 \geq (r_1 \cap r_2) \cup a_1 \geq a_1$, and since $r_1 \rho a_1$ it follows by (E_1) that $((r_1 \cap r_2) \cup a_1) \rho a_1$. The condition of the theorem implies $(r_1 \cap r_2) \rho ((r_1 \cap r_2) \cap a_1) = r_2 \cap a_1$. Similarly, since $r_2 \geq (a_1 \cap r_2) \cup a_2 \geq a_2$ and $r_2 \rho a_2$, $(a_1 \cap r_2) \cup a_2) \bar{\rho} a_2$, and hence $(a_1 \cap r_2) \rho ((a_1 \cap r_2) \cap a_2) = a_1 \cap a_2$. We obtain $(r_1 \cap r_2) \rho (a_1 \cap a_2)$ and $(a_1 \cap r_2) \rho (a_1 \cap a_2)$. Hence it follows by (D_1) that $(r_1 \cap r_2) \rho (a_1 \cap a_2)$. This yields $r(a_1 \cap a_2) \geq r_1 \cap r_2$. Hence $r(a_1) \cap r(a_2) = r(a_1 \cap a_2)$. Since $r(a) \bar{\rho} I$, we have $r(r(a)) = r(a)$, which proves the idempotency of this mapping.

5. Dual relations. On account of the duality of lattices, we develop in this section a theory of dual relations and dual radicals.

Definition 5. A relation σ defined in a lattice M is called a *dual H-relation* if σ satisfies:

(A′) If $a \sigma b$, $a, b \in M$ then $a \leq b$.

(B′) = (B) $a \sigma a$ for every $a \in M$.

(C′) If $a \sigma b$ and $c \leq b$ then $(a \cap c) \sigma c$.

Dually to the non decreasing chain of elements $\{u_\lambda\}$ we define a non increasing chain of radicals $\{l_\lambda\}$ as follows: $l_1(L,\sigma) = \mathrm{Inf}[q; q\,\sigma\, I \text{ in } L]$. $l_\lambda(L,\sigma) = \mathrm{Inf}[l_\nu; \nu < \lambda]$ for limit ordinal λ, and $l_\lambda(L,\sigma) = \mathrm{Inf}[q; q\,\sigma\, l_{\lambda-1} \text{ in } L]$ for non limit ordinal λ. The element $l_\lambda(L,\sigma)$ is called the λ-th *σ-radical of L*. This non increasing chain of radicals terminates at some $l_\tau(L,\sigma)$, that is, there is a minimal τ such that $l_\tau = l_\mu$ for every $\mu \geq \tau$. The element $l_\tau(L,\sigma) = L(L,\sigma)$ will be called the *lower σ-radical* of L.

One readily develops a theory for the lower radical dually to the theory of the upper radical developed in the preceding sections. In particular we refer to a dual H-relation as a *dual R-relation* if it satisfies

(D′) *If $a\,\sigma\, b$ does not hold, then $c\,\bar{\sigma}\, b$ does not hold, then $c\,\bar{\sigma}\, b$ for some $c \in L$, $a \leq c < b$.*

Let ρ be an H-relation defined in M. The relation $\bar{\rho}$ defined in M can be considered as a dual relation in M. We have

THEOREM 5.1. *If ρ is an H-relation in M, then $\bar{\rho}$ is a dual H-relation which satisfies* (D′$_1$).[7] *Furthermore, if the set L of the sublattices of M is the set of the M-intervals, then $\bar{\rho}$ is a dual R-relation, and $u(L,\rho) = r(L,\bar{\rho})$ for every M-interval L.*

Proof. Evidently $\bar{\rho}$ satisfies (A′) and (B′). We proceed in proving (C′). Let $a\,\bar{\rho}\, b$ and $b \geq c$. If $c\,\bar{\rho}(a \cap c)$ is not true, then $p\,\rho(a \cap c)$ for some $c \geq p > c \cap a$. Since ρ satisfies (C), $(a \cup p)\,\rho\, a$. But in view of $b \geq p \cup a \geq a$ and $a\,\bar{\rho}\, b$, it follows that $p \cup a = a$, i. e., $a \geq p$. This together with $c \geq p$ contradicts $p > c \cap a$, and thus (C′) is proved. Condition (D′$_1$) was proved in Lemma 2.3.

Before proceeding with the proof we remark that if $\{L\}$ is the totality of the M-intervals of M, then $a\,\bar{\rho}\, b$ in an M-interval L is the same requirement as $a\,\bar{\rho}\, b$ in the whole lattice M.

We prove now that the upper ρ-radical $u(L,\rho)$ of an M-interval L is the $\bar{\rho}$-radical of L. By Theorem 1.2 it follows that $u(L,\rho)\,\bar{\rho}\, I_L$ in L, and hence $u(L,\rho)\,\bar{\rho}\, I_L$ in M. The interval $[0_L, u(L,\rho)]$ is $\bar{\rho}$-semi simple, for otherwise $q\,\bar{\rho}\, u(L,\rho)$ for some $0_L \leq q < u(L,\rho)$, $q \in L$. Hence Lemma 2.3 implies $q\,\bar{\rho}\, I_L$. This contradicts the minimality of $u(L,\rho)$ proved in Theorem 1.2. Tnis shows that $u(L,\rho) = r(L,\bar{\rho})$, which, by the dual of Theorem 2.1, proves that $\bar{\rho}$ is a dual R-relation.

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[7] (D′$_1$) denotes the dual condition of (D$_1$).

In the light of the preceding theorem, Theorem 1.2 is the dual of Corollary 1.4.

Some of the known relations are at the same time both H-relations and dual H-relations. The following theorem deals with a class of such relations.

THEOREM 5.2. *If ρ is any relation defined in M which satisfies* (A), (B), (E_1) *and the requirement that $(a \cup b)\rho a$ implies $b\,\rho(a \cap b)$, then the relation ρ' defined by $a\,\rho'\,b$ if $b\,\rho\,a$, is a dual H-relation.*

Evidently ρ' satisfies (A') and (B'). Let $a\,\rho'\,b$ and $b \geq c$. Since $b \geq c \cup a \geq a$ and $b\,\rho\,a$, it follows by (E_1) that $(c \cup a)\rho\,a$ and, therefore, $c\,\rho(a \cap c)$. This proves that $(a \cap c)\rho'\,c$, hence the validity of (C').

6. Multiplicative lattices. We conclude this paper with an example which generalizes two known radicals, the lower radical defined by Baer in [1], and the idempotent kernel of rings defined by Levitzki in [2].

By a multiplicative lattice M we mean a complete lattice in which multiplication is defined satisfying

1) For every $a, b \in M$, $ab \in M$ is uniquely defined,

2) $ab \subseteq a \cap b$,

3) $a(b \cup c) = ab \cup ac$; $\qquad (b \cup c)a = ba \cup ca$.

If $a \leq b$, $a \cup b = b$; hence by (3), it follows that $ca \cup cb = cb$, which implies that $ca \leq cb$. In particular $a^2 \leq ab \leq b^2$.

Consider the two relations ζ and its dual ζ' defined in M as follows: $a\,\zeta\,b$ and $b\,\zeta' a$ if $a^2 \leq b \leq a$. Evidently ζ and ζ' satisfy (A), (B) and (A'), (B') respectively. Now let $a\,\zeta\,b$ and $c \geq b$, then $(a \cup c)^2 \leq (a^2 \cup ac \cup ca \cup c^2)$ $\leq b \cup c = c \leq c \cup a$. This proves that ζ is an H-relation. The relation ζ' satisfies (C'), for if $b\,\zeta'\,a$ and $c \leq a$ then $c^2 \leq a^2 \leq b$. Hence, since $c^2 \leq c$, $c^2 \leq b \cap c \leq c$, i. e., $(b \cap c)\zeta'\,c$. Thus ζ' is a dual H-property (One readily verifies that ζ' satisfies also E_1). Hence the upper ζ-radical and the lower ζ'-radical are defined in M.

In case M is the multiplicative lattice of the ideals of an associative ring S, the upper ζ-radical is known as the Baer's lower radical of S ([1]). It is readily seen that the theory of this radical is valid also in non-associative rings. Further properties of this radical will be discussed in Part III of this paper.

The lower ζ'-radical can be approached from a different point of view, due to the fact that the notion of $\overline{\zeta}'$-elements coincides with the idea of the

idempotent elements of M. Indeed, if $a^2 = a$ it is evident that a is a $\overline{\zeta}'$-element in M and conversely, since $a^2 \zeta' a$, the fact that a is a $\overline{\zeta}'$-element must yield $a^2 = a$. Thus we obtain by the dual of Theorem 1. 2 that the lower ζ'-radical of M is the maximal idempotent element of M, and that the interval $[l(M, \zeta'), I]$ does not contain idempotent elements. This result has been obtained for ideals in rings and in semi-groups by J. Levitzki in [2].

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ON THE STRUCTURE OF MULTIPLY TRANSITIVE PERMUTATION GROUPS.* [1]

By T. C. Holyoke.

1. Introduction. Since Jordan [1] first considered the problem of transitive extension of finite transitive permutation groups [2], not much improvement on his work has been made in the way of general results. Zassenhaus [2] obtained results for certain groups of automorphisms of finite geometries. Witt [3, pp. 257-259] specialized Jordan's results to the case of multiply transitive groups, giving his results in such a way that they are still valid if infinite groups are admitted. Miller [4] showed that if a finite group of degree $n > 12$ does not contain the alternating group on its letters and is s-fold transitive, then $s < 3n^{\frac{1}{2}} - 2$. From this result it is possible to infer an upper limit of the number of successive transitive extensions which a given finite group can possess.

In this paper, the theory is extended to cover transitive extensions in general, the finite group appearing in the rôle of a special case for which certain simplifications of a calculational nature can be made. A general transitive extension theory, paralleling Schreier's extension theory, is given. Then the case of extension of a multiply transitive group is taken up in detail, in such a way as to emphasize the important rôle played by certain conjugates of the given group, not only insofar as the existence of transitive extensions is concerned, but also in the actual construction of them. Unlike the results of Jordan and Witt, the process given here is constructive in nature, reducing both the number of permutations to be investigated and the number of tests that need be applied to them, and leading eventually to the realization of the strong dependence of any multiply transitive group on certain associated groups which are exactly doubly and triply transitive.

2. Statement and general solution of the problem. Let there be given a permutation group G fixing a certain letter, say O, and transitive on the remaining letters (not necessarily finite or countable in number). We wish to find a permutation group H which is transitive on all the given letters and

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[1] This paper is based on the author's doctoral dissertation at The Ohio State University.

in which G is the (largest) subgroup fixing the letter O. Such a group H has been referred to as a transitive extension of G [2].

For sections **2** and **3**, the following notation will be used: The letters displaced by G will be denoted by $\{x\}$. The subgroup of G fixing the letter $1 \in \{x\}$ will be denoted by F. For each letter x, a representative $u_x \in G$ will be prescribed which displaces 1 by x. This can be done because G is transitive. Let $u_1 = I$. Then $G = \Sigma_x Fu_x$. A permutation $g \in G$ will be considered the same as

$$\begin{pmatrix} Fu_x \\ Fu_x g \end{pmatrix} \text{ or } \begin{pmatrix} u_x^{-1}F \\ g^{-1}u_x^{-1}F \end{pmatrix},$$

upon identification of letters x, Fu_x and $u_x^{-1}F$. The group G will be given as acting only on letters $\{x\}$, but will be considered the same as that group having the same permutations and fixing any other desired letters. H will denote a transitive extension of G, if one exists. The symbols f, g, and h are generic for elements of F, G, and H respectively, and the symbol u for the chosen coset representatives of F in G. Functions will be denoted by Greek letters, placed to the right of the elements on which they operate, composition of functions by $\circ$; transformation by a permutation p will be denoted by τ_p, and the composite function $\pi^{-1} \circ \gamma \circ \pi$ by γ^π, except that $\tau_p^{-1} \circ \gamma \circ \tau_p$ will be denoted by γ^p. Finally, $g_1(g_2\gamma)$ will be written as $g_1g_2\gamma$, and $(g_1\gamma) \cdot g_2$ as $g_1\gamma g_2$.

Suppose a transitive extension H exists. The following facts, well-known in the finite case, will be of use:

2.1 H contains an element a transposing 0 and 1, hence $a^2 \in F$ [1, p. 31]. If $\{x\}$ is finite, a^2 can be taken as I.

2.2 $a^{-1}Fa = F$. The proof here differs from that in the finite case [1, p. 32] in that $a^{-1}Fa$ could conceivably be a proper subgroup of F. It is not, however, since a^2 induces an automorphism of F.

2.3 For each $g \notin F$ there exist g_1 and $g_2 \notin F$ such that $aga = g_1ag_2$ [1, p. 32].

2.4 $H = G + \Sigma_x Gau_x$, and every h has the form g or the form gau [cf. 3, p. 258].

2.5 $g_1ag_2 = g_3ag_4$ if and only if $g_3^{-1}g_1 \in F$ and $(g_3^{-1}g_2)\tau_a = g_4g_2^{-1}$.

The following theorem serves to formalize the problem of transitive extension in terms of the internal structure of G.

2.6 THEOREM. *A necessary and sufficient condition for the existence of an extension H is that there exist an element q and an automorphism α*

of F, and two single valued mappings β and γ of $G - F$ into itself, satisfying the following relations for every g, g_1, and g_2 not in F and every f, f_1, and f_2 in F:

(a) $g\gamma$ is a u, $q\alpha = q$, $f\alpha \circ \alpha = f\alpha^2 = f\tau_q$;

(b) $\{f_1(f_1^{-1}\alpha g f_2)\beta\}^{-1} g\beta \in F$ and
$$(\{f_1(f_1^{-1}\alpha g f_2)\beta\}^{-1} g\beta)\alpha = (f_1^{-1}\alpha g f_2)\gamma f_2^{-1}\alpha(g\gamma)^{-1};$$

(c) $g\beta(g\gamma f)\beta \in F$ and $\{g\beta(g\gamma f)\beta\}\alpha = gqf\alpha\{(g\gamma f)\gamma\}^{-1}$;

(d) $\{(fg\beta)\beta\}^{-1} qf\alpha g \in F$ and $(\{(fg\beta)\beta\}^{-1} qf\alpha g)\alpha = \{fg\beta\}\gamma(g\gamma)$;

(e) $g_1(g_2\beta) \in F$ if and only if $g_1\gamma g_2 \in F$, in which case
$$g_1\beta q(g_1\gamma g_2)\alpha = q\{g_1 g_2\beta\}\alpha g_2\gamma;$$

(f) For $g_1 g_2\beta$ not in F, $\{(g_1 g_2\beta)\beta\}^{-1} g_1\beta(g_1\gamma g_2)\beta \in F$, and
$$(\{(g_1 g_2\beta)\beta\}^{-1} g_1\beta(g_1\gamma g_2)\beta)\alpha = \{g_1 g_2\beta\}\gamma g_2\gamma\{(g_1\gamma g_2)\gamma\}^{-1}.$$

Proof. Assume the existence of H. By 2.1 and 2.2, H contains an element a such that $a^2 \in F$ and $a^{-1}Fa = F$. Take α and q as τ_a and a^2, respectively. By 2.3, 2.4, and 2.5, for $g \notin F$, $aga = g'au$ (uniquely), with g' and u not in F. Define $g\beta = g'$ and $g\gamma = u$. Relations (b)-(f) now follow from 2.5 (except in the case of (e)) and the associativity of H, considering respectively elements of the following types: $a(f_1^{-1}\alpha g f_2)a$, $agafa$, $afaga$, ag_1ag_2a with $g_1 g_2\beta \in F$, and ag_1ag_2a with $g_1 g_2\beta \notin F$. Conversely, assume that q, α, β, and γ are given and satisfy the prescribed conditions. A suitable group H will be constructed. Consider all objects of types (g) and (g_1, g_2) for all g, g_1, and g_2 in G. Define the equivalence $\sim$: $(g) \sim (g')$ if and only if $g = g'$; $(g) \nsim (g_1, g_2)$; $(g_1, g_2) \sim (g_3, g_4)$ if and only if $g_3^{-1} g_1 \in F$ and $(g_3^{-1} g_1)\alpha = g_4 g_2^{-1}$. Denote equivalence classes by $[g]$ and $[g_1, g_2]$. Let $\bar{H}$ denote the set of equivalence classes. Products are defined as follows:

$$[g_1][g_2] = [g_1 g_2], \quad [g_1][g_2, g_3] = [g_1 g_2, g_3], \quad [g_1, g_2][g_3] = [g_1, g_2 g_3],$$
$$[g_1, g_2][g_3, g_4] = [g_1 q(g_2 g_3)\alpha g_4]$$

if $g_2 g_3 \in F$, and $[g_1, g_2][g_3, g_4] = [g_1(g_2 g_3)\beta, , g_2 g_3)\gamma g_4]$ otherwise. That these products are well-defined follows directly from the definition of equivalence, except in the last two cases. For those cases, suppose $(g_1', g_2') \sim (g_1, g_2)$ and $(g_3', g_4') \sim (g_3, g_4)$. Then there exist f_1 and f_2 such that $g_1' = g_1 f_1$, $g_2' = f_1^{-1}\alpha g_2$, $g_3' = g_3 f_2$, and $g_4' = f_2^{-1}\alpha g_4$. Therefore $g_2 g_3$ belongs to F if and only if $g_2' g_3'$ belongs to F, in which case

$$g_1' q(g_2' g_3')\alpha g_4' = g_1 f_1 q f_1^{-1}\alpha^2(g_2 g_3)\alpha g_4 = g_1 q(g_2 g_3)\alpha g_4,$$

by (a). If $g_2 g_3$ does not belong to F, then the equality of $[g_1, g_2][g_3, g_4]$ and $[g_1', g_2'][g_3', g_4']$ follows from (b) and the definition of equivalence.

The element $[I]$ is a right identity, and right inverses for elements $[g]$ and $[g_1, g_2]$ exist in $[g^{-1}]$ and $[g_2^{-1}, (g_1 g)^{-1}]$ respectively.

That products in $\bar{H}$ are associative follows directly from the definition of products and the associativity of G, except in the case of $[g_1, g_2]([g_3, g_4][g_5, g_6])$. For the various possibilities $g_2 g_3 \in F$ and $g_4 g_5 \in F$; $g_2 g_3 \notin F$ and $g_4 g_5 \in F$; $g_2 g_3 \in F$ and $g_4 g_5 \notin F$; $g_2 g_3 \notin F$ and $g_4 g_5 \notin F$ and $g_2 g_3 (g_4 g_5)\beta \in F$; and finally $g_4 g_5 \notin F$ and $g_2 g_3 \notin F$ and $g_2 g_3 (g_4 g_5)\beta \notin F$, associativity follows from relations (a), (c), (d), (e), and (f) respectively. Hence $\bar{H}$ is a group. Define $\bar{a} = [I, I]$, $\bar{g} = [g]$, $\bar{G}$ the set of all $\bar{g}$, $\bar{F}$ the set of all $\bar{f}$. Then $\bar{G}$ is isomorphic to G, and $[g_1, g_2] = \bar{g}_1 \bar{a} \bar{g}_2$. Also, $\overline{a^{-1} f a} = [q(q\alpha)^{-1} f\alpha] = \overline{f\alpha}$, and $\bar{a}^2 = \bar{q}$. For $\bar{g} \notin \bar{F}$, $\bar{a}\bar{g}\bar{a} = [g\beta, g\gamma] = \overline{g\beta}\ \bar{\alpha}\ \overline{g\gamma}$. $\bar{H} = \bar{G} + \Sigma_x \bar{G}\bar{a}\bar{u}_x$, for, given $\bar{h} = [g_1, g_2]$, $\bar{h}$ can also be written in the form $[g, u]$, where $g_2 \in Fu$. Thus $h = \bar{g}\bar{a}\bar{u}$. The cosets are clearly distinct. It remains to be shown that $\bar{H}$ has a faithful permutation representation H on letters O and $\{x\}$ in which the image of $\bar{a}$ displaces O and the image of $\bar{g}$ is exactly the permutation g with the added fixed letter O. Such a representation is given by $\bar{h} \to \begin{pmatrix} \bar{G} \cdots \bar{G}\bar{a}\bar{u} \cdots \\ \bar{G}h \cdots \bar{G}\bar{a}\bar{u}h \cdots \end{pmatrix}$, under the correspondence $O \rightleftarrows \bar{G}$, $x \rightleftarrows \bar{G}\bar{a}\bar{u}_x$. Here $\bar{a} \to \begin{pmatrix} \bar{G} & \bar{G}\bar{a} \cdots \\ \bar{G}\bar{a} & \bar{G}\bar{a}^2 \cdots \end{pmatrix} = \begin{pmatrix} 0 & 1 \cdots \\ 1 & 0 \cdots \end{pmatrix}$, and $\bar{g} \to \begin{pmatrix} \bar{G} \cdots \bar{G}\bar{a}\bar{u} \cdots \\ \bar{G} \cdots \bar{G}\bar{a}\bar{u}\bar{g} \cdots \end{pmatrix}$. But $g = \begin{pmatrix} Fu \\ Fug \end{pmatrix}$, and $\bar{G}\bar{a}\bar{u}\bar{g} = \bar{G}\bar{a}\bar{u}_x$ if and only if $Fug = Fu_x$. Thus H has the properties asserted in the theorem.

2.8 *Definition.* If q, α, β, and γ satisfy the conditions of Theorem 2.7, $(q, \alpha, \beta, \gamma)$ will be called an *embedding system.*

Notation. The "bar" notation of Theorem 2.7 will now be discarded. Let the mappings ϕ and ψ be defined as follows: $g\phi$ is the coset representative u for g, and $g\psi$ the element of F such that $g = g\psi g\phi$.

2.9 *Definition.* Two systems $(q, \alpha, \beta, \gamma)$ and $(q^*, \alpha^*, \beta^*, \gamma^*)$ are *similar* if there exist an element f and an automorphism π of G mapping F onto itself such that $g\gamma^* = (g\gamma^\pi f\alpha^\pi)\phi$ for all $g \notin F$. Two similar systems for which π can be taken as the identity are *equivalent.*

2.10 *Definition.* Two transitive groups of permutations on the same set of letters are *permutation-isomorphic* if there exists a permutation of those letters transforming one onto the other.

2.11 THEOREM. *Two transitive extensions of G are permutation-isomorphic if and only if the corresponding embedding systems are similar, and are identical if and only if the corresponding systems are equivalent,*

*under the natural correspondence of letters Gau with Ga*u and G with G, respectively.*

Proof. Suppose H and H^* are permutation-isomorphic. Of the permutations transforming H onto H^*, at least one fixes the letters 0 and 1, since H^* is doubly transitive. Let p be such a permutation. Then $G\tau_p = G$ and $F\tau_p = F$. Also, $a\tau_p \in Fa^*$, so that $a^* = fp^{-1}ap$. Now

$$g\beta^* a^* g\gamma^* = a^* g a^* = fp^{-1}apgfp^{-1}ap = fp^{-1}ag\tau_{p^{-1}}af\tau_{p^{-1}} \circ \alpha p$$
$$= fg\beta^p a\tau_p (g\gamma^p f\alpha^p)\psi(g\gamma^p f\alpha^p)\phi = [fg\beta^p(g\gamma^p f\alpha^p)\psi \circ (\alpha^{-1})^p f^{-1}]a^*(g\gamma^p f\alpha^p)\phi,$$

so that $g\gamma^*$ has the desired form, with $\pi = \tau_p$.

Conversely, suppose two systems and an element f are given such that $g\gamma^* = (g\gamma^\pi f\alpha^\pi)\phi$, where π is an automorphism of G mapping F onto itself. Recall that in H^* the letter Ga^*u, and in H the letter Gau, correspond to the letter Fu as given in G. In the following discussion, all representatives u are different from the identity. There exists a permutation p fixing letters 0 and 1 such that $\pi = \tau_p$ [3, p. 259], in particular, $p = \begin{pmatrix} Fu \\ Fu\pi \end{pmatrix}$. To show that $p^{-1}Hp = H^*$, it will suffice now, by 2.4, to show that $a^* = fa\tau_p$.

$$a\tau_p = (01)\begin{pmatrix} (Gau)p \\ (Gau\gamma)p \end{pmatrix} = (01)\begin{pmatrix} (Fu)p \\ (Fu\gamma)p \end{pmatrix} = (01)\begin{pmatrix} Fu\pi \\ Fu\gamma \circ \pi \end{pmatrix}$$
$$= (01)\begin{pmatrix} Gau\pi \\ Gau\gamma \circ \pi \end{pmatrix} = (01)\begin{pmatrix} Ga^*u' \\ Ga^*u'\gamma^\pi \end{pmatrix}.$$

Hence

$$fa\tau_p = a\tau_p f\alpha^\pi = (01)\begin{pmatrix} Ga^*u \\ Ga^*u\gamma^\pi \end{pmatrix} \cdot (0)(1)\begin{pmatrix} Ga^*u\gamma^\pi \\ Ga^*u\gamma^\pi f\alpha^\pi \end{pmatrix} = (01)\begin{pmatrix} Ga^*u \\ Ga^*u\gamma^\pi f\alpha^\pi \end{pmatrix}$$
$$= (01)\begin{pmatrix} Ga^*u \\ Ga^*(u\gamma^\pi f\alpha^\pi)\phi \end{pmatrix} = (01)\begin{pmatrix} Ga^*u \\ Ga^*u\gamma^* \end{pmatrix} = a^*.$$

If π is the identity automorphism, then p can be taken as the identity permutation, and $fa = a^*$, hence H and H^* are the same.

2.12 COROLLARY. The relations *similar* and *equivalent* of 2.9 are equivalences.

The importance of the concept of similarity of systems is brought out in section **4**.

The connection between the mappings β and γ and the permutation a can be partially expressed in terms of permutations. Define the permutations b and c (on the cosets of F in G as letters) by the following: $(F)b = (F)c = F$; $(gF)b = g\beta F$ and $(Fg)c = Fg\gamma$ for $g \notin F$. If b and c are considered as permutations of the letters of G, they fix the letter 1, and

$b^{-1} = c = (01) \cdot a$. Furthermore, $G_{\tau_c} \not\subset G$ unless G contains all finite even permutations of its letters.

2..13 *Remark.* G has no transitive extension if it does not contain all finite even permutations of its letters and if its normalizer in the group of all permutations of letters $\{x\} - 1$ contains the normalizer of F on those letters.

3. Some groups having no transitive extensions. The following theorems indicate two infinite classes of finite simply transitive groups for which transitive extensions do not exist.

3.1 THEOREM. *Let G be a transitive permutation group on letters $1, 2, \cdots, n, n+1, n+2, \cdots, n+n$ such that the subgroup F fixing the letter 1 also fixes $n+1$, and has two multiply transitive constituents on the sets $2, \cdots, n$ and $n+2, \cdots, n+n$ respectively, so constituted that in each element f, the fixed letters are congruent* (mod n) *in pairs, and congruent letters are displaced by congruent letters; or let G be permutation-isomorphic to such a group. Then G has no transitive extension.*

Proof. Permutation-isomorphic groups have or fail to have transitive extensions together; so the group G can be taken as described. F contains an elements f of order two, having an even number of transpositions, say $2j$. Then $2j \leqq n-1$. Suppose an extension H exists. H contains an element a interchanging 0 and 1, and conjugate to f [1, p. 31], therefore having $2j-1$ transpositions among letters $2, 3, \cdots, 2n$. Now $a^{-1}Fa = F$ (2.2), hence a fixes the letter $n+1$ and either permutes the letters of each transitive set among themselves, or must interchange the sets entirely. The latter case is impossible, for a would have $n-1$ transpositions among $2, 3, \cdots, 2n$. Therefore it may be supposed that a takes each transitive set into itself. Then a has the transposition (ij) among letters $2, \cdots, n$ if and only if it has the transposition $(n+i, n+j)$, since different permutations induce different automorphisms of a multiply transitive group. This contradicts the fact that $2j-1$ is odd. Hence no such group H exists.

3.2 COROLLARY. *If the symmetric group of degree $n > 4$ is represented as a group G of permutations on the cosets of one of its alternating subgroups of degree $n-1$, then G has no transitive extension.*

3.3 THEOREM.[2] *If G is a transitive group $(\neq I)$ of odd order and of degree $n \equiv 1 \pmod 4$, then G has no transitive extension.*

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[2] The author is indebted to M. Hall for this theorem.

Proof. Suppose an extension H exists. The order of H is divisible by 2 but not by 4; hence H has a Sylow subgroup of order 2. Since a subgroup of order 2 is in the center of its normalizer, H contains a normal subgroup H^* of index 2 in H [5, p. 327]. G, being maximal in H, is not contained in H^*. Therefore $G \cap H^*$ has index 2 in G, yielding a contradiction.

4. Multiply transitive groups. The notation will now be changed in accordance with the following: for $k > 1$, let G_k represent a k-fold transitive group on letters $\{x\}$, $0, 1, \cdots, k-1$, not containing an element consisting of a cycle of length 2 or 3; and for $i = 0, 1, \cdots, k-1$, let G_i be the subgroup of G_{i+1} fixing the letter i. The letter added in the attempt to find transitive extensions of G_k will be called k. Let $G_k^{(\rho)}$, for variable index ρ, represent all the distinct transitive extensions of G_{k-1} other than G_k.

4.1 *Remark.* In G_i $(i = 2, \cdots, k)$, there exists an element a_i transposing letters $i-1$ and $i-2$, permuting letters $\{x\}$ among themselves, and fixing the rest [cf. 1, p. 32, and 3, p. 257]. Let elements similar to a_k be chosen also in each $G_k^{(\rho)}$ and designated by $a_k^{(\rho)}$. All these elements are permutable with G_0 [3, p. 257].

4.2 *Definition.* Let $a_{k+1}^{(\rho)} = a_k^{(\rho)\tau_{a_k}} \cdot (k, k-1, k-2)$.

4.3 *Remark.* If adjunction of various elements $a_{k+1}^{(\rho)}$ to G_k produces transitive extensions of G_k, then all such extensions are distinct.

Proof. If $a_{k+1}^{(\rho)}$ and $a_{k+1}^{(\rho')}$ yield the same extension, then

$$a_{k+1}^{(\rho)} \varepsilon G_0 a_{k+1}^{(\rho')},$$

from which it follows that $a_k^{(\rho)} \varepsilon G_0 a_k^{(\rho')}$, and $G_k^{(\rho)} = G_k^{(\rho')}$, i. e. $\rho = \rho'$.

4.4 THEOREM. *In order for a particular $a_{k+1}^{(\rho)}$ to yield a transitive extension of G_k, it is necessary and sufficient that $(a_k^{(\rho)} a_k)^3 \varepsilon G_0$ and $a_{k+1}^{(\rho)} G_{k-1} a_{k+1}^{(\rho)} = G_{k-1}$. For $k > 2$, these conditions are equivalent to the two conditions $(a_k^{(\rho)} a_k)^3 \varepsilon G_0$ and $(a_{k+1}^{(\rho)} a_{k-1})^2 \varepsilon G_0$ (in Theorem 4.11, it is shown that for $k > 3$, the single condition $(a_{k+1}^{(\rho)} a_{k-1})^2 \varepsilon G_0$ is sufficient).*

Proof. Since $(a_k a_{k+1}^{(\rho)})^3 = (a_k^{(\rho)} a_k)^3$, this theorem follows from a theorem of Jordan [1, p. 33] and Witt [3, p. 257].

4.5 THEOREM. *Each transitive extension of G_k can be obtained by adjoining to it one of the elements $a_{k+1}^{(\rho)}$, in which case the corresponding $G_k^{(\rho)}$ (uniquely determined, as follows from 4.3) is permutation-isomorphic to G_k, the element $a_{k+1}^{(\rho)} \cdot (k, k-1)$ transforming G_{k-1} into itself and a_k into some element of the coset $G_0 a_k^{(\rho)}$.*

Proof. Let G_{k+1} be a transitive extension of G_k, a_{k+1} an element of G_{k+1} which interchanges k and $k-1$, and fixes $k-2, \cdots, 0$. Define $a_k' = a_k\tau_{a_{k+1}} \cdot (k, k-1, k-2)$. Then $a_k' \notin G_k$, otherwise $(k, k-1, k-2)$ would belong to G_{k+1}, in which case G_k would also contain a cycle of length 3, since G_{k+1} is triply transitive. G_k and $G_{k-1} \cup \{a_k'\}$ are permutation-isomorphic, since the element $a_{k+1} \cdot (k, k-1)$ transforms G_{k-1} into itself and a_k into a_k'. $G_{k-1} \cup \{a_k'\}$ is an extension of G_{k-1} and is therefore one of the $G_k^{(\rho)}$. Now $a_k^{(\rho)} = fa_k'$ for some element $f \in G_0$, and

$$\begin{aligned} a_{k+1}^{(\rho)} &= a_k^{(\rho)}\tau_{a_k} \cdot (k, k-1, k-2) \\ &= a_k^{-1}fa_{k+1}a_ka_{k+1} \cdot (k, k-1, k-2) \cdot a_k \cdot (k, k-1, k-2) \\ &= a_k^{-1}fa_{k+1}^{-1}a_ka_{k+1}a_k \in G_{k+1}, \end{aligned}$$

hence $G_{k+1} = G_k \cup \{a_{k+1}^{(\rho)}\}$ (2. 4), and $a_{k+1}^{(\rho)} = f_1a_{k+1}$ for some element $f_1 \in G_0$. Since all the a's are permutable with G_0, the final assertion of the theorem follows.

4. 6 COROLLARY. *If the number of distinct extensions of G_{k-1} which are permutation-isomorphic to G_k is a finite number t, then the number of extensions of G_k is less than t; in particular, if a finite simply transitive group G of degree $n > 3$ has at most t permutation-isomorphic extensions, then G cannot be extended to a $(t+1)$-fold transitive group of degree $n+t+1$.*

4. 7 *Definition.* If every $a_{k+1}^{(\rho)}$ produces a transitive extension of G_k, it will be said that G_k *can be extended in every possible way.*

4. 8 LEMMA. *If, for $k > 3$, G_k and G_k' are two extensions of G_{k-1}, then there exist permutations s_1, s_2, s_3, and s_4 on letters $\{x\}$ such that $G_{k-2}^{(i)} = G_{k-3} \cup \{s_i \cdot (k-3, k-4)\}$ $(i = 1, 2, 3, 4)$ are transitive extensions of G_{k-3}, $G_{k-1}^{(j)} = G_{k-2}^{(1)} \cup \{s_j\tau_{s_1} \cdot (k-2, k-3)\}$ $(j = 2, 3, 4)$ are transitive extensions of G_{k-2} $(= G_{k-2}^{(1)})$, and $G_k^{(m)} = G_{k-1}^{(2)} \cup \{s_m\tau_{s_2} \circ \tau_{s_1} \cdot (k-1, k-2)\}$ $(m = 3, 4)$ are the two given extensions G_k and G_k', respectively of G_{k-1} $(= G_{k-1}^{(2)})$.*

Proof. This follows from successive application of 4. 2-4. 5.

Let τ_i represent τ_{s_i} throughout the remainder of this paper.

4. 9 LEMMA. $s_i^2 \equiv (s_2s_m)^3 \equiv I \pmod{G_0}$,

$$s_m\tau_2 \circ \tau_1 \equiv s_m\tau_2, \text{ and } s_4\tau_3 \circ \tau_2 \circ \tau_1 \equiv s_4\tau_3 \circ \tau_2,$$

$$(i = 1, 2, 3, 4; m = 3, 4).$$

Proof. $s_i^2 \equiv I$, by (2.1). $(s_2 s_m)^3 \equiv I$, since, by 4.4,

$$\{s_2\tau_1 \cdot (k-2, k-3) \cdot s_m\tau_1 \cdot (k-2, k-3)\}^3 \equiv I.$$

By Theorem 4.4, $I \equiv (s_m\tau_2 \circ \tau_1 s_1)^2 \equiv s_m\tau_2 \circ \tau_1 s_m \tau_2$, from which it follows that $s_m\tau_2 \circ \tau_1 \equiv s_m{}^{-1}\tau_2 \equiv s_m\tau_2$. Finally,

$$\begin{aligned} s_4\tau_3 \circ \tau_2 &\equiv s_3\tau_2 s_4\tau_2 s_3\tau_2 \\ &\equiv s_3\tau_2 \circ \tau_1 s_4\tau_2 \circ \tau_1 s_3\tau_2 \circ \tau_1 \equiv (s_3 s_4 s_3)\tau_2 \circ \tau_1 \\ &\equiv s_4\tau_3 \circ \tau_2\tau_1. \end{aligned}$$

4.10 LEMMA. *$(s_4\tau_3 \circ \tau_2 s_2 \tau_1)^2 \equiv I$, if and only if $(s_4\tau_3 \circ \tau_2 s_2)^2 \equiv I$, and if these relations hold, then $(s_4\tau_3 \circ \tau_2 s_3 \tau_2)^3 \equiv I$.*

Proof. Lemma 4.9 will be used here repeatedly. Assume

$$(s_4\tau_3 \circ \tau_2 s_2 \tau_1)^2 \equiv I.$$

Then

$$I \equiv (s_4\tau_3 \circ \tau_2 \circ \tau_1 s_2 \tau_1)^2 \equiv (s_4\tau_3 \circ \tau_2 s_2)^2 \tau_1,$$

whence $(s_4\tau_3 \circ \tau_2 s_2)^2 \equiv I$. Assume that $(s_4\tau_3 \circ \tau_2 s_2)^2 \equiv I$. Then

$$(s_4\tau_3 \circ \tau_2 s_2 \tau_1)^2 \equiv (s_4\tau_3 \circ \tau_2 \circ \tau_1 s_2 \tau_1)^2 \equiv (s_4\tau_3 \circ \tau_2 s_2)^2 \tau_1 \equiv I.$$

Also $s_4\tau_3 \circ \tau_2 \equiv s_4{}^{-1}\tau_3 \equiv s_4\tau_3$, and

$$(s_4\tau_3 \circ \tau_2 s_3 \tau_2)^3 \equiv (s_4\tau_3 s_3\tau_2)^3 \equiv (s_4\tau_3 s_2\tau_3)^3 \equiv (s_4 s_2)^3 \tau_3 \equiv I.$$

4.11 THEOREM. *In order for $a_{k+1}{}^{(\rho)}$ (cf. 4.2 and 4.4) to yield a transitive extension of G_k when $k > 3$, it is sufficient that $(a_{k+1}{}^{(\rho)} a_{k-1})^2 \in G_0$.*

Proof. Let the groups G_k and G_k' of 4.8-4.10 correspond to the groups G_k and $G_k{}^{(\rho)}$ of this theorem, respectively, $s_3\tau_2 \circ \tau_1 \cdot (k-1, k-2)$ and $s_4\tau_2 \circ \tau_1 \cdot (k-1, k-2)$ to a_k and $a_k{}^{(\rho)}$, and $s_2\tau_1 \cdot (k-2, k-3)$ to a_{k-1}. Then $a_{k+1}{}^{(\rho)} = s_4\tau_3 \circ \tau_2 \circ \tau_1 \cdot (k, k-1)$, and the theorem is seen to follow from 4.9, 4.10, and 4.4.

4.12 THEOREM. *If, for $k > 3$, every $(k-1)$-fold transitive extension of G_{k-3} can be extended in every possible way (cf. 4.7), then all the extensions of G_{k-1} are permutation-isomorphic and can themselves be extended in every possible way.*

Proof. If G_k and $G_k{}^{(\rho)}$ are two extensions of G_{k-1}, it is required to show that $a_{k+1}{}^{(\rho)}$ yields a transitive extension of G_k. Let the same correspondence be set up as in the proof of the previous theorem. Then the following groups are $(k-1)$-fold transitive extension of G_{k-3}:

$$\{G_{k-3}, s_1(k-3, k-4), s_2\tau_1 \cdot (k-2, k-3)\}\ (= G_{k-1}),$$

$$\{G_{k-3}, s_1(k-3, k-4), s_3\tau_1(k-2, k-3)\},$$

and

$$\{G_{k-3}, s_1(k-3, k-4), s_4\tau_1(k-2, k-3)\}.$$

Since $G_{k-3}\tau_2^{s_1} = G_{k-3}$ (Theorem 4. 4), it follows that

$$\{G_{k-3}, s_1(k-3, k-4), s_3\tau_1(k-2, k-3)\}\tau_2^{s_1}$$
$$= \{G_{k-3}, s_2(k-3, k-4), s_3\tau_2(k-2, k-3)\}$$

and

$$\{G_{k-3}, s_1(k-3, k-4), s_4\tau_1(k-2, k-3)\}\tau_2^{s_1}$$
$$= \{G_{k-3}, s_2(k-3, k-4), s_4\tau_2(k-2, k-3)\}$$

are also $(k-1)$-fold transitive extensions of G_{k-3}. The assumption that each such group can be extended in every possible way leads to the conclusion that $I \equiv (s_3^{-1}\tau_2 s_4\tau_2 s_3\tau_2 s_2)^2 \equiv (s_4\tau_3 \circ \tau_2 s_2)^2 \pmod{G_0}$; and the desired result now follows from 4. 10 and 4. 11.

4. 13 *Remark.* It is easy to show that each permutation of the letters $\{x\}$ which, with the adjunction of a suitable transposition, is used to obtain an extension of G_1 having degree of transitivity greater than three, has already been used similarly in obtaining some extension of G_1 which is exactly triply transitive. This fact seems to lend weight to the conjecture that there are only a finite number of triply transitive groups which have transitive extensions.

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ON THE DETERMINATION OF MULTIPLY CONNECTED DOMAINS OF AN ELASTIC PLANE BODY, BOUNDED BY FREE BOUNDARIES WITH CONSTANT TANGENTIAL STRESSES.* [1]

By C. Arf.

1. Formulation of the problem.

B. Consider the closed curves L of the z plane which are boundaries of multiply connected domains C of multiplicity m, such that there exists a state of stress of C which satisfies the following conditions.

B_1. The stress tensor $(\sigma_x, \sigma_y, \tau)$ satisfies the condition $\sigma_x + \sigma_y = 4\alpha$ with a given constant α throughout C and on L.

B_2. The stress tensor $(\sigma_x, \sigma_y, \tau)$ has determined finite or infinite values everywhere in C and on the boundary L, with a possible exception at $z=x+iy=\infty$. More precisely the analytic function $\Gamma(z) = (\sigma_x - \sigma_y - 2i\tau)/4\alpha$ has no singularities other than poles in C and on the boundary L except at the point $z=\infty$

B_3. The boundary L is a free boundary for the state of stress $(\sigma_x, \sigma_y, \tau)$.

B_4. $\Gamma(z)$ has only a finite number of poles in C and on L. Besides these we shall require that the following geometrical conditions are satisfied.

B_5. The domain C does not contain the point $z=\infty$. But the point $z=\infty$ might be a point of the boundary L. For a variable point z of the domain which tends toward ∞ continuously, $\arg z$ has limits which are the numbers $\phi_1, \phi_2, \cdots, \phi_n$ $(0 \leq \phi_1 < \phi_2 < \cdots < \phi_n < 2\pi)$.

B_6. To every ϕ_j is associated a set of separated intervals

$$(a_1^{(j)}, b_1^{(j)}), (a_2^{(j)}, b_2^{(j)}), \cdots, (a_{\nu_j}^{(j)}, b_{\nu_j}^{(j)}),$$

which are described by the expression $\lim_{z\to\infty} \Im z \exp(-i\phi_j)$, where $z \in C$ is supposed to go to ∞ continuously and $\lim_{z\to\infty} \arg z = \phi_j$.

* Received February 20, 1951.

[1] I wish to thank Dr. R. C. Roberts for helping me prepare the English version of this paper.

B_7. L has only a finite number of points of inflection and a finite number of cuspidal points.

It is clear that these conditions require that the curve L have the form shown in (fig. 1) and that the isolated exterior forces which are applied to the elastic plane body $(C + L)$ are like those shown on that figure.

The purpose of this paper is to determine the domains C and the stress tensors σ_x, σ_y, τ which satisfy the conditions above.

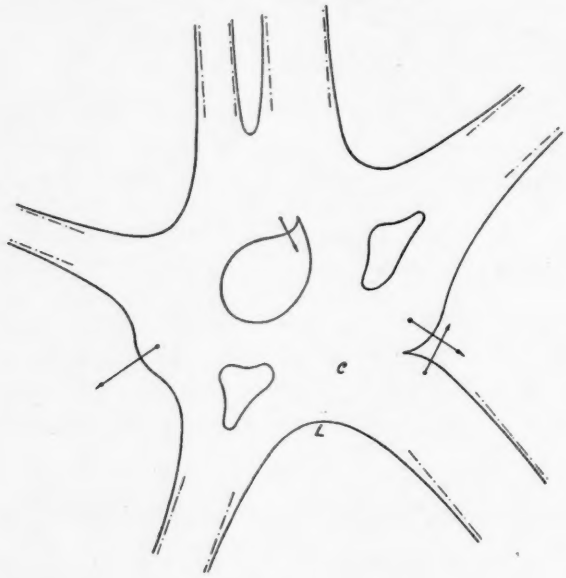

FIG. 1.

2. In our former publications [2] we have shown that a state of stress for which $\sigma_x + \sigma_y = 4\alpha$ is a constant has free boundaries if and only if it is defined in the following manner except for the case where the stresses are constants throughout the plane.

Let $\chi(t)$ be an analytic function which takes real values on some parts of the real axis of the T-plane. We put

$$z = \Phi(\zeta) = i\int \chi(i(\zeta - 1)/(\zeta + 1))\, d\zeta/\zeta^2 \tag{2.1}$$

The function $z = \Phi(\zeta)$ defines a conformal mapping (locally) of the ζ-plane onto the z-plane. At the point $z = \Phi(\zeta)$ we consider the tensor

[2] C. Arf, "Sur les frontières libres à tensions constantes d'un milieu élastique plan en équilibre," *Notes communiquées au VI^e congrès international de mécanique appliquée*, 1946.

$$\begin{aligned} \sigma_x &= 2\alpha(1+\Re\zeta^2), \\ \sigma_y &= 2\alpha(1-\Re\zeta^2), \\ \tau &= -2\alpha\Im\zeta^2, \end{aligned} \tag{2.2}$$

which defines a state of elastic stress of the z-plane. The point

$$z = \Phi((1-it)/(1+it)) \tag{2.3}$$

describes a free boundary of this state of stress, when t varies on the part of the real axis of the plane where $\chi(t)$ is real. If ϕ denotes the angle of the tangent to the boundary with the x axis, the center of curvature of L at the corresponding point z of the boundary will be

$$z + i\chi(tg\tfrac{1}{2}\phi)e^{i\phi}, \tag{2.4}$$

and we will have

$$z = \Phi((1-itg\tfrac{1}{2}\phi)/(1+itg\tfrac{1}{2}\phi)) = \Phi(e^{-i\phi}), \qquad \zeta = e^{-i\phi} \tag{2.5}$$

3. We consider now the function

$$Z = \zeta^2 = (\sigma_x - \sigma_y - 2i\tau)/4\alpha = \Gamma(z) \tag{3.1}$$

which transforms conformally our domain C and its boundary L into a Riemann surface $\mathfrak{L}$ and its boundary k respectively. The point $Z = \Gamma(z)$ tends toward the point $Z = \exp(-2i\phi_l)$ when z tends toward $z = \infty$ continuously under the condition $\lim_{z\to\infty} \arg z = \phi_l$. The set $\mathfrak{L} + k = E$, in which we include the points $\exp(-2i\phi_l)$, is then a closed domain of a certain Riemann surface over the Z plane. We consider now the symmetric surface $\mathfrak{L}^s$ of $\mathfrak{L}$ with respect to the boundary k of E. The set $\mathfrak{L}^s + \mathfrak{L} + k = R$ will then be a closed Riemann surface. Let us consider a connected part L_1 of L. According to our assumptions, L has only a finite number of cuspidal points and a finite number of points of inflection. Therefore the variation of the angle ϕ of the tangent to L, when the point z describes the connected part L_1, will be such that $Z = \exp(-2i\phi)$ will draw a finite number of arcs of finite length of the unit circle $|Z| = 1$. The boundary k will then be constituted by m such closed arcs. If we now take into account that $Z = \Gamma(z)$ can take the value $Z = \infty$ only a finite number of times, or that E has only a finite number of points at infinity. We can conclude that E and therefore R has only a finite number of sheets. *R is therefore an algebraic Riemann surface of genus $m-1$ and has only a finite number of points of ramification.*

Let M_{Z_0} be a point of E different from the images of the point $z = \infty$.

Let $t=(Z-Z_0)^{1/\mu}$ (or $t=Z^{-1/\mu}$ if $Z_0=\infty$) be the local uniformization parameter of R at this point M_{Z_0}. We consider the inverse function $z=\Lambda(Z)$ of $Z=\Gamma(z)$ at this point M_{Z_0}. Because of the fact that the mapping $z=\Lambda(Z)$ of E into $C+L$ is conformal, we shall have

$$\Lambda(Z)=a_0+a_1t+\cdots+a_nt^n+\cdots \qquad \text{with } a_1\neq 0, \tag{3.2}$$

and therefore

$$\Lambda'(Z)=1/\mu t^{-\mu+1}(a_1+2a_2t+\cdots+na_nt^{n-1}+\cdots) \text{ for } Z_0 \text{ finite}, \tag{3.3}$$

$$\Lambda'(Z)=\nu/\mu t^{\mu+1}(a_1+2a_2t+\cdots+na_nt^{n-1}+\cdots) \text{ for } Z_0=\infty. \tag{3.3'}$$

From this we can deduce the following results:

An ordinary point of E which does not correspond to a point at infinity on L is an ordinary point of $\Lambda'(Z)$.

A point of ramification of order μ of E which does not correspond to a point at infinity on L is a pole of order $\mu-1$ of $\Lambda'(Z)$, if its affix is finite.

A point M_∞ of ramification of order μ on E is a zero of order $\mu+1$ of $\Lambda'(Z)$.

The function $\Lambda(Z)$ is obviously equal to $\Lambda(Z)=\Phi(Z^{\frac{1}{2}})$. We have therefore

$$\Lambda'(Z)=\Phi'(Z^{\frac{1}{2}})/2Z^{\frac{1}{2}}=i\chi(i(Z^{\frac{1}{2}}-1)/(Z^{\frac{1}{2}}+1))/2ZZ^{\frac{1}{2}}, \tag{3.4}$$

or

$$-\chi(iZ^{\frac{1}{2}}-1)/(Z^{\frac{1}{2}}+1))=2iZZ^{\frac{1}{2}}\Lambda'(Z). \tag{3.5}$$

Thus the function $iZZ^{\frac{1}{2}}\Lambda'(Z)$ has to be real on the boundary k of E. We can therefore extend this function on $\mathfrak{L}^s$ by the formula

$$iZZ^{\frac{1}{2}}\Lambda'(Z)=\overline{iZ^*Z^{*\frac{1}{2}}\Lambda'(Z^*)} \tag{3.6}$$

where Z^* is the symmetrical point of Z with respect to k. One can write this as follows:

$$\Lambda'(Z)=-\overline{Z^{*2}\Lambda'(Z^*)} \tag{3.7}$$

This relation enables us to extend the function $\Lambda'(Z)$ on $\mathfrak{L}^s$. Our results about the singularities of $\Lambda'(Z)$ on $\mathfrak{L}$ will then have the following consequences on $\mathfrak{L}^s$.

A point of ramification of order μ on $\mathfrak{L}^s$ is a pole of order $\mu-1$ of $\Lambda'(Z)$, if this point has a finite affix different from zero.

A point of ramification of order μ on $\mathfrak{L}^s$ whose affix is zero, is a pole of order $2\mu - 1$ of $\Lambda'(Z)$.

A point at infinity on $\mathfrak{L}^s$ is a zero of order $2\mu + 1$ of $\Lambda'(Z)$, if this point has an order of ramification μ on $\mathfrak{L}^s$.

We can also show in exactly the same way as we have done in our former work [3] that a *point P of R which corresponds to a point at infinity on L is a pole of order μ for $\Lambda'(Z)$, if this point P is a point of ramification of order μ on R.*

In the following parts of the paper we shall study only the special case $m = 2$. The case $m > 2$ will be studied in a forthcoming paper.

4. Let us denote by L_1 the exterior boundary of our domain C, and by L_2 its interior boundary. L_2 will have no point at infinity.

For $m = 2$ the genus of R is 1, and we can represent Z and $\Lambda'(Z)$ as elliptic functions $H(T)$ and $R(T)$ of a variable $T = t + it'$.

Let ω_1, ω_2 be the fundamental periods of these functions, and let us consider the corresponding σ-function of Weierstrass. It follows from the fact that R is symmetrical with respect to k that to each zero a of order μ of $Z = H(T)$ corresponds a pole b of order μ of $H(T)$. The function $H(T)$ has then the following form

$$H(T) = A \prod_{j=1}^{\nu} (\sigma(T - a_j)/\sigma(T - b_j))^{\mu_j}, \tag{4.1}$$

where the points $a_1, a_2, \cdots, a_\nu$; $b_1, b_2, \cdots, b_\nu$ are different from each other and satisfy the condition

$$\sum_{j=1}^{\nu} \mu_j a_j = \sum_{j=1}^{\nu} \mu_j b_j. \tag{4.2}$$

We consider now a point M on R and the symmetrical point M^s with respect to k. Let T and T^s be the corresponding points of the T-plane.

$$T \to \bar{T}^s \tag{4.3}$$

is then a conformal mapping of the finite T-plane on itself. This mapping is therefore of the form

$$\bar{T}^s = \alpha T + \beta \qquad \alpha, \beta \text{ constants.} \tag{4.4}$$

[3] C. Arf, "Sur la détermination des états d'équilibres d'un milieu élastique plan admettant des frontières libres à tensions constantes," *Revue de la Faculté des Sciences de l'Université d'Istanbul*, tome XII, Serie A, 1947.

From $(M^s)^s = M$ we deduce $(T^s)^s \equiv T \mod(\omega_1, \omega_2)$, which means

$$(4.5)\qquad \begin{aligned} &\bar{\alpha}(\alpha T + \beta) + \bar{\beta} \equiv T && \mod(\omega_1, \omega_2), \\ &|\alpha| = 1, \quad \bar{\alpha}\beta + \bar{\beta} \equiv 0 && \mod(\omega_1, \omega_2). \end{aligned}$$

By a preliminary substitution of the form $T \to aT + b$ we can make $\alpha = 1$ and β real.[4] We can then either choose as one of our periods ω_1, ω_2 (ω_1 for example) a real number, and we have $\beta = \omega_1/2$ or $\beta = 0$. Besides this, to every set of congruent points $T + m_1\omega_1 + m_2\omega_2$ must correspond a set $T^s + m_1'\omega_1 + m_2'\omega_2$ of congruent points. This means that we must have $\bar{T} + m_1\omega_1 + m_2\bar{\omega}_2 + \beta = \bar{T} + \beta + m_1'\omega_1 + m_2'\omega_2$. Therefore the conjugates of periods must be periods too. The fundamental periods are then either of the form $\omega_1 = \omega$, $\omega_2 = i\omega'$, or of the form $\omega_1 = 2\omega$, $\omega_2 = \omega + i\omega'$ (ω, ω' real). If $\beta = \omega_1/2$, the images of k in the T-plane are the lines $T = \bar{T} + \omega_1/2 + m_1\omega_1 + m_2\omega_2$, or

$$(4.6)\qquad 2\Im T = m_2\omega', \qquad m_1 + \tfrac{1}{2} = 0, \qquad \text{for } \omega_1 = \omega, \omega_2 = i\omega',$$

$$(4.7)\qquad 2\Im T = m_2\omega', \qquad m_2 + 2m_1 + 1 = 0 \qquad \text{for } \omega_1 = 2\omega, \omega_2 = \omega + i\omega'.$$

The equations (4.6) have no solutions, while the lines which are solutions of (4.7) are deduced from one of them by the translations $n\omega_2$. In the case $\beta = \omega_1/2$ the boundary k will have therefore either no point at all or consist only of one piece.

We must have therefore $\beta = 0$. The images of k in the T-planes are then the lines

$$(4.8)\qquad 2\Im T = m_2\omega', \qquad m_1 = 0, \qquad \text{for } \omega_1 = \omega, \omega_2 = i\omega',$$

$$(4.9)\qquad 2\Im T = m_2\omega', \qquad m_2 + 2m_1 = 0, \qquad \text{for } \omega_1 = 2\omega, \omega_2 = \omega + i\omega'.$$

The lines which are solutions of (4.9) are obtained from one of them by the translations $m\omega_2$. The boundary k will consist in that case of a curve of one piece only. *We shall therefore have to consider only the case*

$$(4.10)\qquad \omega_1 = \omega, \qquad \omega_2 = i\omega' \qquad T^s = \bar{T}. \qquad (\omega, \omega' \text{ real})$$

The fundamental periods being of this form, the functions

$$\sigma(T) = T\Pi'\{(1 - T/(m\omega + im'\omega'))\exp(T/(m\omega + im'\omega') + T^2/2(m\omega + im'\omega')^2)\},$$

$$\zeta(T) = 1/T + \Sigma'(1/(T - m\omega - im'\omega') + 1/(m\omega + im'\omega') + T/(m\omega + im'\omega')^2),$$

[4] Such a substitution carries the coefficients α and β into $\alpha a/\bar{a}$, $\beta + (\bar{a}b - a\bar{b})/\bar{a}$.

will be real on the real axis, and we shall have $\sigma(\bar{T}) = \sigma(T)$, $\zeta(\bar{T}) = \zeta(T)$. The points of affix zero and the points at infinity on R being symmetrical with respect to k, the corresponding points a_j and b_j of the T plane will have to satisfy the condition

$$b_j \equiv \bar{a}_j, \qquad \mod(\omega, i\omega'). \tag{4.11}$$

We consider now the functions $R(T) = \Lambda'(Z) = \Lambda'(H(T))$ and $F(T) = \Lambda(Z) = \Lambda(H(T))$. Since the functions $R(T)$ and $H(T)$ are elliptic of periods $\omega, i\omega'$, the function

$$F'(T) = \Lambda'(Z) \cdot H'(T) = R(T)H'(T) \tag{4.12}$$

is also elliptic of periods $\omega, i\omega'$. The function $\Lambda'(Z)$ has to satisfy the condition

$$\Lambda'(Z) = -\overline{\Lambda'(Z)z^3},$$

at every point Z of k (3.7). The function $F'(T)$ will have therefore to satisfy the condition

$$F'(T)/H'(T) = -\overline{H(T)}^3 \overline{F'(T)}/\overline{H'(T)} \tag{4.13}$$

for all values of T for which $\Im T = 0$ or $\omega'/2$. But for these values of T we have

$$Z\bar{Z} = H(T) \cdot \overline{H(T)} = 1, \qquad H'(T)\overline{H(T)} + \overline{H'(T)}H(T) = 0.$$

The condition (4.13) can therefore be written as follows:

$$H(T) = \overline{F'(T)}/F'(T), \qquad \text{for } \Im T = 0, \omega'/2. \tag{4.14}$$

If we put $\underline{F}'(T) = \overline{F'(\bar{T})}$, we can deduce from (4.14) and from the fact that $F'(T)$ is an elliptic function of periods $(\omega, i\omega')$ that

$$H(T) = \underline{F}'(T)/F'(T), \tag{4.15}$$

for all values of T. Reciprocally one can deduce from (4.15) the relations (4.14), $H(T) \cdot \overline{H(T)} = 1$ and (4.13) for $\Im T = 0$ or $\omega'/2$. The function $z = \Lambda(Z) = F(T)$ maps one of the strips

$$0 < \Im T < \omega'/2, \qquad -\omega'/2 < \Im T < 0$$

on our domain C. But by a substitution of the form $T \to aT + b$ ($a = \pm 1$, $b = 0, i\omega'/2$) we can make this function map the strip $0 < \Im T < \omega'/2$ on the domain C and the line $\Im T = 0$ on the exterior boundary L_1 of C. This mapping must obviously be conformal in that strip $0 < \Im T < \omega'/2$. *The*

zeros and the poles of $F'(T)$ are therefore in the strips which can be obtained from $-\omega'/2 \leq \Im T \leq 0$ by translations of magnitudes $m\omega'$

Let us consider a point T_0 on $\Im T = 0$ or $\Im T = \omega'/2$. If the image $F(T_0) = \Lambda(H(T_0))$ of this point on the boundary L is different from ∞ and from a cuspidal point of L, then the point $Z_0 = H(T_0)$ of k is a pole of order $\mu - 1$ of $\Lambda'(Z)$, where μ is the order of ramification of this point on R. The point T_0 is then a zero of order $\mu - 1$ of $H'(T)$, and the function $F'(T) = \Lambda'(H(T)) \cdot H'(T)$ will be regular at the point T_0. If the image $F(T_0)$ is a point at infinity on L_1 (in which case $\Im T_0 = 0$), the point $Z_0 = H(T_0)$ on k is a pole of order μ of $\Lambda'(Z) = \Lambda'(H(T))$, and the function $F'(T) = \Lambda'(H(T)) \cdot H'(T)$ has a simple pole at T_0.

The function $F'(T)$ has therefore no singularities on $\Im T = \omega'/2$, and its only singularities on $\Im T = 0$ are poles of order 1.

The zeros of $F'(T)$ on the lines $\Im T = 0$, $\Im T = \omega'/2$ have also to be of order 1. In fact, if T_0 is a zero of $F'(T)$ on one of these lines, then the image of the neighborhood of T_0 in $0 < \Im T < \omega'/2$ by the mapping $z = F(T)$ will have more than one sheet unless $F''(T_0) \neq 0$. Under this condition the neighborhood of $F(T_0)$ in C will have to be of one of the following three forms [5]

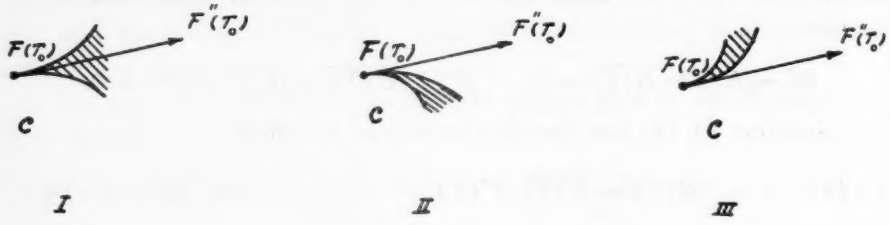

FIG. 2.

But it can be easily shown that the cuspidal point $F(T_0)$ of L has one of these three forms if and only if

$$(4.16)\quad \left.\begin{array}{ll} \Im(F^{(\nu)}(T_0)/F''(T_0)) > 0 & \text{for } \Im T_0 = 0 \\ \Im(F^{(\nu)}(T_0)/F''(T_0)) < 0 & \text{for } \Im T_0 = \omega'/2 \end{array}\right\} \text{for odd values of } \nu,$$

where $F^{(\nu)}(T)$ is the lowest order derivative of $F(T)$ such that

$$\Im(F^{(\nu)}(T_0)/F''(T_0)) \neq 0.$$

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[5] C. Arf, "Sur la détermination des états d'équilibres d'un milieu élastique plan admettant des frontières libres à tensions constantes," *Revue de la Faculté des Sciences de l'Université d'Istanbul*, tome XII, Serie A, 1947.

Finally, the function $z = F(T) = \int F'(T)\,dT$ has to be chosen in such a way that the domain $0 \leq \Im T \leq \omega'/2$, $0 \leq \Re T < \omega$ is mapped on a doubly connected closed domain of the z plane. For this it is necessary to have

$$F(T+\omega) - F(T) = \int_0^\omega F'(T+t)\,dt = 0 \tag{4.17}$$

for all values of T with $0 < \Im T < \omega'/2$. In order to make this condition explicit we consider the following representation of $F'(T)$:

$$F'(T) = c + \sum_{j=1}^{n} \{A_j\zeta(T-\beta_j) + A'_j\zeta'(T-\beta_j) + \cdots + A_j^{(\mu_j-1)}\zeta^{(\mu_j-1)}(T-\beta_j)\}, \tag{4.18}$$

where

$$A_j(T-\beta_j)^{-1} - A'_j(T-\beta_j)^{-2} + 2A''_j(T-\beta_j)^{-3} - \cdots + (-1)^{\mu_j-1}(\mu_j-1)\,!A_j^{(\mu_j-1)}(T-\beta_j)^{-\mu_j} \tag{4.19}$$

is the principal part of $F'(T)$ at the pole β_j, c is a suitable constant, and where the coefficients A_j have to satisfy the condition

$$\sum_{j=1}^{n} A_j = 0. \tag{4.20}$$

We put now this representation of $F'(T)$ in (4.17), and we obtain the following condition for the coefficients A_j, A'_j, and the constant c:

$$0 = \int_0^\omega F'(T+t)\,dt = c\omega + \sum_{j=1}^{n} \{A_j \int_0^\omega \zeta(T+t-\beta_j)\,dt + A'_j(\zeta(T+\omega-\beta_j) - \zeta(T-\beta_j))\}. \tag{4.21}$$

Using the known relations

$$\zeta(T-\beta_j+\omega) - \zeta(T-\beta_j) = 2\zeta(\omega/2) = \eta,$$

$$\int_0^\omega \zeta(T-\beta_j+t)\,dt = \log(\sigma(T-\beta_j+\omega)/\sigma(T-\beta_j)),$$

$$= \eta(T-\beta_j+\omega/2) + i\pi,$$

we can write this condition as follows

$$c\omega - \eta(\sum_{j=1}^{n} A_j\beta_j - \sum_{j=1}^{n} A'_j) = 0. \tag{4.22}$$

We consider again a pole $T = \beta$ of $F'(T)$ on the line $\Im T = 0$. As we have noticed before, this pole is of order one, and in the neighborhood of β we have

$$(4.23) \qquad F(T) = A\int\zeta(T-\beta)dT + R_1(T)$$
$$= A\log\sigma(T-\beta) + R_1(T) = A\log(T-\beta) + R(T),$$

where $R_1(T)$, $R(T)$ are regular in the neighborhood of $T=\beta$. If one follows the variation of this function in the neighborhood of β in the strip $0 \leq \Im T \leq \omega'/2$, one sees that the function $F(T)$ maps such a neighborhood on a domain like the one shown in figure 3.

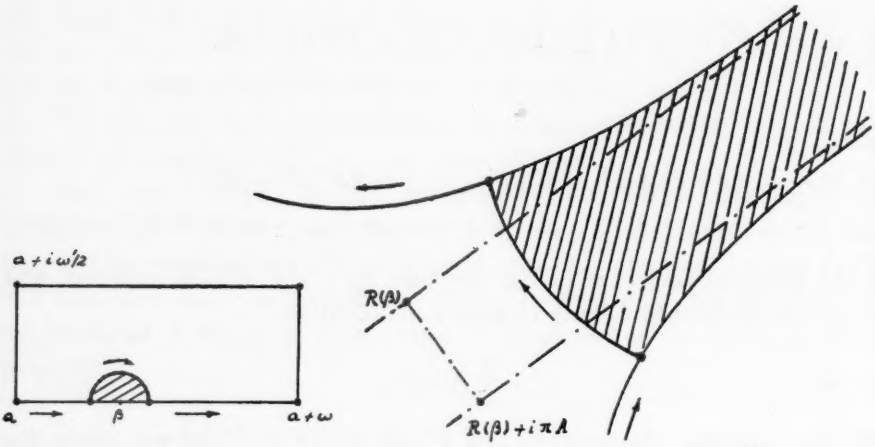

FIG. 3.

The neighborhood of $T=\beta$ corresponds to an infinite strip of c whose direction is given by the coefficient $-A$ of the principal part of $F'(T)$ at $T=\beta$. The thickness of this strip is obviously $|A|\pi$. If therefore $\beta_1, \beta_2, \cdots, \beta_h$ is the increasing sequence of the poles of $F'(T)$ in $0 \leqq T < \omega$, and $A_1 = -f_1\exp(i\phi_1)$, $A_2 = -f_2\exp(i\phi_2), \cdots, A_h = -f_h\exp(i\phi_h)$ are the corresponding residues of $F'(T)$, we must have

$$0 \leqq \phi_2 - \phi_1 \leqq \phi_3 - \phi_1 \leqq \cdots \leqq \phi_h - \phi_1 \leqq 2\pi,$$

in order that the corresponding strips of c do not overlap each other.

Besides this we must also require that $F'(T)$ be such that none of the closed curves

$$z = F(i(\omega'/2 - 0) + t), \qquad 0 \leq t < \omega,$$

$$z = F(+i0 + t), \qquad 0 \leq t < \omega,$$

cuts itself.

Conversely let a function

$$F'(T) = \omega^{-1}\eta\Big(\sum_{j=1}^{k}(A_j\gamma_j - A'_j) - \sum_{j=1}^{h}\beta_j f_j \exp(i\phi_j)\Big) - \sum_{j=1}^{h} f_j \exp(i\phi_j)\zeta(T-\beta_j)$$
$$+ \sum_{j=1}^{k}\{A_j\zeta(T-\gamma_j) + A'_j\zeta'(T-\gamma_j) + \cdots + A_j^{(\mu_j-1)}\zeta^{(\mu_j-1)}(T-\gamma_j)\}$$

be given which satisfies the following six conditions:

I. $$\sum_{j=1}^{k} A_j - \sum_{j=1}^{h} f_j \exp(i\phi_j) = 0,$$

II. $$0 \leq \beta_1 < \beta_2 < \cdots < \beta_h < \omega,$$

III. $$f_j > 0, \qquad 0 \leq \phi_2 - \phi_1 < \phi_3 - \phi_1 < \cdots < \phi_h - \phi_1 \leq 2\pi$$

IV. $$-\omega'/2 < \Im\gamma_j < 0, \qquad 0 \leq \Re\gamma_j < \omega.$$

V. *The function $F'(T)$ has no zero in $0 < \Im T < \omega'/2$, and its zeros α_l on the lines $\Im T = 0$, $\Im T = \omega'/2$ are all simple and satisfy the condition*

$$\Im(F^{(\nu)}(\alpha_l)/F''(\alpha_l)) > 0 \quad \textit{for } \Im\alpha_l = 0 \textit{ and } \nu \textit{ odd},$$

$$\Im(F^{(\nu)}(\alpha_l)/F''(\alpha_l)) < 0 \quad \textit{for } \Im\alpha_l = \omega'/2 \textit{ and } \nu \textit{ odd},$$

where $F^{(\nu)}(T)$ is the first derivative of $F(T)$ for which $\Im(F^{(\nu)}(\alpha_l)/F''(\alpha_l)) \neq 0$.

VI. *None of the curves*

$$z = F(+i0 + t), \qquad 0 \leq t < \omega,$$

$$z = F(i(\omega'/2 - 0) + t), \qquad 0 \leq t < \omega,$$

cuts itself.

The function

$$(4.24) \qquad z = F(T) = \omega^{-1}\eta\Big(\sum_{j=1}^{k}(A_j\gamma_j - A'_j) - \sum_{j=1}^{n} f_j\beta_j \exp(i\phi_j)\Big)T$$
$$- \sum_{j=1}^{h} f_j \exp(i\phi_j)\log(\sigma(T-\beta_j))$$
$$+ \sum_{j=1}^{k}\{A_j \log(\sigma(T-\gamma_j)) + A'_j\zeta(T-\gamma_j)$$
$$+ \cdots + A_j^{(\mu_j-1)}\zeta^{(\mu_j-2)}(T-\gamma_j)\}$$

maps the domain $0 \leq \Im T \leq \omega'/2$, $0 \leq \Re T < \omega$ on a doubly connected simple domain C which satisfies the conditions B_5, B_6, B_7, *and the distribution of stress given by the formula*

$$(4.25) \qquad x+iy=z=F(T),$$

$$\sigma_x=2\alpha(1+\Re(\underline{F}'(T)/F'(T))), \qquad \sigma_y=2\alpha(1-\Re(\underline{F}'(T)/F'(T))),$$

$$\tau=-2\alpha\Im(\underline{F}'(T)/F'(T)), \qquad \underline{F}'(T)=\overline{F'(\bar{T})}$$

satisfies the conditions B_1, B_2, B_3, B_4.

5. In this section we shall compute the following quantities.

a) The resultant $X+iY$ of the forces which are exerted on the boundary (b) of a small body containing the point $z_j=F(\bar{\gamma}_j)$ where $\underline{F}'(T)/F'(T)=(\sigma_x-\sigma_y-2i\tau)/4\alpha$ becomes infinite,

b) The resulting momentum of these forces with respect to the point $F(\bar{\gamma}_j)$,

c) The displacement $u+iv$ which corresponds to the state of stress given by (4.25).

a)
$$X+iY=\oint_{(b)}(\tau dx-\sigma_x dy)+i\oint_{(b)}(\sigma_y dx-\tau dy)$$

$$=2\alpha\oint_{(b)}-\Im(\underline{F}'(T)/F'(T))dx-(1+\Re(\underline{F}'(T)/F'(T)))dy,$$

$$+2\alpha i\oint_{(b)}(1-\Re(\underline{F}'(T)/F'(T)))dx+\Im(\underline{F}'(T)/F'(T))dy,$$

$$(5.1) \qquad X+iY=2\alpha i\oint_{(b)}dz-\overline{(\underline{F}'(T)/F'(T))dz}=-2\alpha i\oint_{(b_T)}F'(\bar{T})d\bar{T},$$

where (b_T) designates the image of (b) in the T-plane. Since

$$A_j(T-\gamma_j)^{-1}-A'_j(T-\gamma_j)^{-2}+\cdots+(-1)^{\mu_j-1}(\mu_j-1)!A_j^{(\mu_j-1)}(T-\gamma_j)^{-\mu_j}$$

is the principal part of $F'(T)$ at the point γ_j, the value of this integral is $X_j+iY_j=-2\alpha i\overline{(2\pi i\bar{A}_j)}$, so that

$$(5.2) \qquad X_j+iY_j=-4\pi\alpha A_j.$$

The thickness of the strip which corresponds to the pole β_j being πf_j, the resultant of the forces exerted at infinity on that strip is $-4\pi\alpha f_j\exp(i\phi_j)$. We remark therefore that the condition

$$(I) \qquad \sum_{j=1}^{k}f_j\exp(i\phi_j)-\sum_{j=1}^{k}A_j=0,$$

which expresses the fact that the function $F'(T)$ is elliptic, also expresses the fact that the resultant of the external forces acting on (C), is zero.

b) The resulting moment of the forces which act on the boundary (b) is given by

$$M_j = \oint_{(b)} (x - \Re F(\bar{\gamma}_j))(\sigma_y dx - \tau dy) - (y - \Im F(\bar{\gamma}_j))(\tau dx - \sigma_x dy),$$

$$= 2\alpha \oint_{(b)} (1 - \Re(\underline{F}'(T)/F'(T)))(x - \Re F(\bar{\gamma}_j))dx + \Im(F'(T)/F'(T)(x - \Re F(\bar{\gamma}_j))dy$$
$$+ \Im(\underline{F}'(T)/F'(T))(y - \Im F(\bar{\gamma}_j)dx + (1 + \Re(\underline{F}'(T)/F'(T)))(y - \Im F(\bar{\gamma}_j))dy.$$

$$= 2\alpha \oint_{(b)} \tfrac{1}{2} d \mid z - F(\bar{\gamma}_j)\mid^2 - \Re(\underline{F}''(T)/F'(T))\Re((z - F(\bar{\gamma}_j))dz)$$
$$+ \Im(\underline{F}'(T)/F'(T))\Im((z - F(\bar{\gamma}_j))dz),$$

$$= 2\alpha \oint_{(b)} \tfrac{1}{2} d \mid z - F(\bar{\gamma}_j)\mid^2 - \Re[(\underline{F}'(T)/F'(T))(z - F(\bar{\gamma}_j))dz],$$

$$= 2\alpha \oint_{(b_T)} - \Re[\underline{F}'(T)(F(T) - F(\bar{\gamma}_j))dT],$$

$$= 4\alpha\pi\Im[\bar{A}'_j F'(\bar{\gamma}_j) - \bar{A}''_j F''(\bar{\gamma}_j) + \cdots + (-1)^{\mu_j}\bar{A}_j^{(\mu_j - 1)} F^{(\mu_j - 1)}(\bar{\gamma}_j)],$$

and the moment with respect to the origin is given by

$$M_j^0 = M_j - 4\pi\alpha\Im\bar{A}_j F(\bar{\gamma}_j).$$

The moment $\mathfrak{M}_j^0$ of the forces which are acting on the strip which corresponds to the pole β_j of $F'(T)$ is easy to compute. One finds

$$\mathfrak{M}_j^0 = -4\pi\alpha[\Im(F(T) + f_j \exp(i\phi_j)\log\sigma(T - \beta_j))_{T=\beta_j} - \tfrac{1}{2}\pi f_j^2].$$

If we compute now the sum

$$\sum_{j=1}^{k} \mathfrak{M}_j^0 - \sum_{j=1}^{k} M_j^0,$$

we obtain zero.

c) From the usual form

$$\begin{bmatrix} \sigma_x & \tau \\ \tau & \sigma_y \end{bmatrix} = \begin{bmatrix} \mu u_x + \tfrac{1}{2}\lambda(u_x + v_y) & \tfrac{1}{2}\mu(u_y + v_x) \\ \tfrac{1}{2}\mu(u_y + v_x) & \mu v_y + \tfrac{1}{2}\lambda(u_x + v_y) \end{bmatrix}$$

of Hooke's law one can deduce easily the following expression of the displacement vector which corresponds to the distribution of stress

$$\sigma_x = 2\alpha(1 + \Re(\underline{F}'(T)/F'(T))), \qquad \sigma_y = 2\alpha(1 - \Re\underline{F}'(T)/F'(T)),$$
$$\tau = -2\alpha\Im(\underline{F}'(T)/F'(T));$$
$$u + iv = (2\alpha/\mu)(z2\mu/(\lambda + \mu) + \int\underline{F}'(T)dT).$$

This displacement is obviously uniform in our domain C if and only if $A_1 = A_2 = \cdots = A_k = 0$, and

$$0 = \int_0^\omega \underline{F}'(T + t)dt \equiv \eta(-\sum_{j=1}^{k} \bar{A}'_j - \sum_{j=1}^{h} f_j \exp(-i\phi_j)\beta_j$$
$$+ \eta(\sum_{j=1}^{k} A'_j + \sum_{j=1}^{h} f_j \exp(-i\phi_j)\beta_j)$$

which is in fact identically satisfied.

6. In this section we shall consider the special case where the quantity $\underline{F}'(T)/F'(T) = (\sigma_x - \sigma_y - 2i\tau)/4\alpha$ is finite everywhere in C. In that case the function $\underline{F}'(T)$ can have no pole in the strip $0 < \Im T \leq \omega'/2$. $F'(T)$ is therefore of the form

$$(6.1) \qquad F'(T) = -\sum_{j=1}^{h} f_j \exp(i\phi_j)\,(\zeta(T - \beta_j) + \eta\beta_j/\omega).$$

The function $F'(T)$ must not have all its zeros on the lines $\Im T = n\omega'$, $\Im T = \omega'/2 + m\omega'$; for in this case the function $H(T) = \underline{F}'(T)/F'(T)$ would be a constant. Indeed, if

$$(6.2) \qquad F'(T) = A\prod_{l=1}^{h}(\sigma(\tau - \alpha_l)/\sigma(T - \beta_l)) \quad \text{with} \quad \sum_{l=1}^{h}\alpha_l - \sum_{l=1}^{n}\beta_l = 0,$$

where the points α_l are on the lines $\Im T = n\omega'/2$, then $\bar{\alpha}_l = \alpha_l + n_l i\omega'$, with $\sum_{l=1}^{h} n_l = 0$, and

$$(6.3) \qquad \underline{F}'(T) = \bar{A}\prod_{l=1}^{h}(\sigma(T - \alpha_l)/\sigma(T - \beta_l))\exp(-\sum_{l=1}^{h} n_l\eta'(T - \alpha_l - in_l\omega'/2)),$$
$$\underline{F}'(T) = (\bar{A}/A)F'(T)\exp(\sum_{l=1}^{h} n_l\eta'(\alpha_l + in_l\omega'/2)).$$

Now we are going to show that the function $F'(T)$ *must have no zero at all on the lines $\Im T = n\omega'/2$.*

Let us suppose that α_1, for example, is on one of these lines $\Im T = n\omega'/2$; α_1 is then a simple zero of $F'(T)$, and according to V. has to satisfy the condition

$$F''(\alpha_1) \neq 0 \qquad \Im(F'''(\alpha_1)/F''(\alpha_1)) \geqq 0 \quad \text{for } n \text{ even},$$

(6.4)

$$F''(\alpha_1) \neq 0 \qquad \Im(F'''(\alpha_1)/F''(\alpha_1)) \leqq 0 \quad \text{for } n \text{ odd}.$$

We have

$$F''(T) = F'(T)\Big(\sum_{l=1}^{h} (\sigma'(T-\alpha_l)/\sigma(T-\alpha_l)) - \sum_{l=1}^{h} (\sigma'(T-\beta_l)/\sigma(T-\beta_l))\Big), \tag{6.5}$$

$$F'''(T) = F''(T)\Big(\sum_{l=1}^{h} (\sigma'(T-\alpha_l)/\sigma(T-\alpha_l)) - \sum_{l=1}^{h} (\sigma'(T-\beta_l)/\sigma(T-\beta_l))\Big)$$
$$+ F'(T)\Big(\sum_{l=1}^{h} (\sigma'(T-\alpha_l)/\sigma(T-\alpha_l))' - \sum_{l=1}^{h} (\sigma'(T-\beta_l)/\sigma(T-\beta_l))'\Big), \tag{6.6}$$

and

$$F'''(T)/F''(T) = F''(T)/F'(T) + \Big(\sum_{l=1}^{h} (\sigma'(T-\alpha_l)/\sigma(T-\alpha_l))' - \sum_{l=1}^{h} (\sigma'(T-\beta_l)/\sigma(T-\beta_l))'\Big)F'(T)/F''(T). \tag{6.7}$$

From this formula we get

$$F'''(\alpha_1)/F''(\alpha_1) = [F''(T)/F'(T) + (\sigma'(T-\alpha_1)/\sigma(T-\alpha_1))'F'(T)/F''(T)]_{T=\alpha_1} \tag{6.8}$$

But from (6.5) we can deduce the following expression of $F''(T)/F'(T)$ in the neighborhood of $T = \alpha_1$:

$$F''(T)/F'(T) = 1/\sigma(T-\alpha_1) + A_0 + A_1\sigma(T-\alpha_1) + \cdots \tag{6.9}$$

with

$$A_0 = \sum_{l=2}^{h} (\sigma'(\alpha_1-\alpha_l)/\sigma(\alpha_1-\alpha_l)) - \sum_{l=1}^{h} (\sigma'(\alpha_1-\beta_l)/\sigma(\alpha_1-\beta_l)). \tag{6.10}$$

We have therefore

$$F'''(\alpha_1)/F''(\alpha_1) = [A_0 + 1/\sigma(T-\alpha_1) + (\sigma'(T-\alpha_1)/\sigma(T-\alpha_1))'\sigma(T-\alpha_1)(1 - A_0\sigma(T-\alpha_1) + \cdots)]_{T=\alpha_1}. \tag{6.11}$$

We have also

$$\sigma'(T-\alpha_1)/\sigma(T-\alpha_1) = 1/\sigma(T-\alpha_1) - \tfrac{1}{3}c_2\sigma^3(T-\alpha_1) + \cdots, \tag{6.12}$$

$$(\sigma'(T-\alpha_1)/\sigma(T-\alpha_1))' = -1/\sigma^2(T-\alpha_1) + \tfrac{2}{3}c_2\sigma^2(T-\alpha_1) + \cdots, \tag{6.13}$$

and therefore

$$F'''(\alpha_1)/F''(\alpha_1) = 2A_0 = 2\Big(\sum_{l=2}^{h}\zeta(\alpha_1-\alpha_l)+\sum_{l=1}^{h}\zeta(\alpha_1-\beta_l)\Big), \tag{6.14}$$

$$\Im(F'''(\alpha_1)/F''(\alpha_1)) = 2\Im\Big(\sum_{l=2}^{h}\zeta(\alpha_1-\alpha_l)-\sum_{l=1}^{h}\zeta(\alpha_1-\beta_l)\Big). \tag{6.14^1}$$

We have seen that the zeros $\alpha_1, \alpha_2, \cdots, \alpha_h$ of $F'(T)$ are congruent to some points $\alpha'_1, \alpha'_2, \cdots, \alpha'_h$ of the strip $-\omega'/2 \leq \Im T \leq 0$, and that at least one of these points α'_j is in the strip $-\omega'/2 < \Im T < 0$. Using this and the relations $\sum_{l=1}^{h}\alpha_l - \sum_{l=1}^{h}\beta_l = 0$, $\Im\beta_l = 0$, we can write $\alpha_l = \alpha'_l + in_l\omega'$, with

$$\Im\sum_{l=1}^{h}\alpha_l = \sum_{l=1}^{h}\Im\alpha'_l + \omega'\sum_{l=1}^{h}n_l = 0,$$

and

$$\Im(F'''(\alpha_1)/F''(\alpha_1)) = 2\Im\Big[\sum_{l=2}^{h}(\zeta(\alpha'_1-\alpha'_l)-2(n_l-n_1)\zeta(i\omega'/2)) - \sum_{l=1}^{h}(\zeta(\alpha'_1-\beta_l)+2n_1\zeta(i\omega'/2))\Big],$$

$$\Im(F'''(\alpha_1)/F''(\alpha_1)) = 2\Im\Big[\sum_{l=2}^{h}(\zeta(\alpha'_1-\alpha'_l)-2((\alpha'_1-\alpha'_l)/i\omega')\zeta(i\omega'/2)) - \sum_{l=1}^{h}(\zeta(\alpha'_1-\beta_l)-2((\alpha'_1-\beta_l)/i\omega')\zeta(i\omega'/2)\Big],$$

$$\Im(F'''(\alpha_1)/F''(\alpha_1)) = (-4/\omega')\Re\Big[\sum_{l=2}^{h}(\zeta(\alpha'_1-\alpha'_l)i\omega'/2 - (\alpha'_1-\alpha'_l)\zeta(i\omega'/2)) - \sum_{l=1}^{h}(\zeta(\alpha'_1-\beta_l)i\omega'/2-(\alpha'_1-\beta_l)\zeta(i\omega'/2)\Big]. \tag{6.15}$$

We have now

$$\begin{aligned}&\Re(\zeta(T)i\omega'/2 - T\zeta(i\omega'/2)) \geq 0, \quad \text{for } n\omega' \leq \Im T \leq n\omega' + \omega'/2,\\ &\Re(\zeta(T)i\omega'/2 - T\zeta(i\omega'/2)) \leq 0, \quad \text{for } n\omega' + \omega'/2 \leq \Im T \leq (n+1)\omega',\end{aligned} \tag{6.16}$$

where the equalities hold if and only if $\Im T = m\omega'/2$.[6] In (6.15) the points $\alpha'_1 - \beta_l$ are, according to our hypothesis, all on the line $\Im T = 0$ or $\Im T = -\omega'/2$, and the points $\alpha'_1 - \alpha'_l$ are in the strip

$$\begin{aligned}0 \leq \Im T \leq \omega'/2, &\qquad \text{for } \Im\alpha'_1 = 0,\\ -\omega'/2 \leq \Im T \leq 0, &\qquad \text{for } \Im\alpha'_1 = -\omega'/2,\end{aligned}$$

[6] See Appendix.

at least one of them being inside of the corresponding strip. From this and from (6. 15) we obtain the inequalities

$$\Im(F'''(\alpha_1)/F''(\alpha_1)) < 0 \qquad \text{for } \Im\alpha'_1 = 0$$

$$\Im(F'''(\alpha_1)/F''(\alpha_1)) > 0 \qquad \text{for } \Im\alpha'_1 = -\omega'/2$$

which contradict the condition (6. 4).

The function $F'(T)$ must have therefore all its zeros inside the strips $n'\omega + \omega'/2 < \Im T < (n+1)\omega'$ and the curve L has no cuspidal point.

We will show now that *the curve L has no point of inflexion.*

Let us consider a point $z = F(t + i\epsilon\omega'/2)$ of L (t real and $\epsilon = 0$ or 1). The angle of the tangent to the boundary at this point, with the x axis being ϕ, we have

$$\underline{F}'(T)/F'(T) = H(T) = \exp(-2i\phi), \qquad T = t + i\epsilon\omega'/2,$$

and therefore

$$-2i\phi = \log \underline{F}'(T) - \log F'(T)$$

$$= \log \bar{A} - \log A + \sum_{l=1}^{h} (\log \sigma(T - \bar{\alpha}_l) - \log \sigma(T - \alpha_l)).$$

It is then enough to show that the derivative of this expression does not vanish for real values of $t = T - i\epsilon\omega'/2$.

$$-2i\phi_t = \sum_{l=1}^{h} (\zeta(t + i\epsilon\omega'/2 - \bar{\alpha}_l) - \zeta(t + i\epsilon\omega'/2 - \alpha_l)),$$

$$-2\phi_t = \sum_{l=1}^{h} \Im(\zeta(t + i\epsilon\omega'/2 - \bar{\alpha}'_l) + 2n_l\zeta(\omega' i/2)$$

$$- \zeta(t + i\epsilon\omega'/2 - \alpha'_l) + 2n_l\zeta(\omega' i/2)),$$

$$(6.17) \quad \phi_t = [\sum_{l=1}^{h} \Re(\zeta(t + i\omega'\epsilon/2 - \bar{\alpha}'_l)\omega' i/2 - (t + i\omega'\epsilon/2 - \bar{\alpha}'_l)\zeta(\omega' i/2))$$

$$- \sum_{l=1}^{h} \Re(\zeta(t + i\omega'\epsilon/2 - \alpha'_l)\omega' i/2 - (t + i\omega'\epsilon/2 - \alpha'_l)\zeta(\omega' i/2))]1/\omega'.$$

The points α'_l being all in the strip $-\omega'/2 < \Im T < 0$, the points $t + i\omega'\epsilon/2 - \bar{\alpha}'_l$ are all in the strip $-\omega'/2 < \Im T < 0$ for $\epsilon = 0$ and $0 < \Im T < \omega'/2$ for $\epsilon = 1$, while the points $t + i\omega'\epsilon/2 - \alpha'_l$ are all in

$0 < \Im T < \omega'/2$ for $\epsilon = 0$ and $\omega'/2 < \Im T < \omega'$ for $\epsilon = 1$. We obtain then by the inequalities (6.16)

$$\phi_t < 0 \quad \text{for } \epsilon = 0; \qquad \phi_t > 0 \quad \text{for } \epsilon = 1. \tag{6.18}$$

Let us follow now the variation of ϕ when the point $z = F(T)$ describes one of the curves L_1, L_2. We have

$$\begin{aligned}\phi = (1/\omega')\int \sum_{l=1}^{h} \Re(\zeta(t + i\omega'\epsilon/2 - \bar{\alpha}'_l)\omega' i/2 \\ - (t + i\omega'\epsilon/2 - \bar{\alpha}'_l)\zeta(\omega' i/2))\,dt \\ - \sum_{l=1}^{h} \Re(\zeta(t + i\omega'\epsilon/2 - \alpha'_l)\omega' i/2 - (t + i\omega'\epsilon/2 - \alpha'_l)\zeta(\omega' i/2))dt.\end{aligned} \tag{6.19}$$

The variation of ϕ along these lines is therefore

$$\begin{aligned}\delta\phi = (1/\omega')\Re \sum_{l=1}^{h} \Big(\int_{\tau^*_l}^{\tau^*_l+\omega} (\zeta(\tau)\omega' i/2 - \tau\zeta(\omega' i/2))d\tau \\ - \int_{\tau_l}^{\tau_l+\omega} (\zeta(\tau)\omega' i/2 - \tau\zeta(\omega' i/2))d\tau\Big),\end{aligned} \tag{6.20}$$

with $\tau_l = i\omega'\epsilon/2 - \alpha'_l$, $\tau^*_l = i\omega'\epsilon/2 - \bar{\alpha}'_l$, where the integrals are taken along the lines parallel to the real axis of the τ plane. By integrating the function $\zeta(\tau)i\omega'/2 - \tau\zeta(i\omega'/2)$ on the parallelogram $(-\omega/2 + i\omega'/2, \tau_0, \tau_0 + \omega, \omega/2 + i\omega'/2)$, we obtain

$$\begin{aligned}\int_{\tau_0}^{\tau_0+\omega} (\zeta(\tau)\omega' i/2 - \tau\zeta(\omega' i/2))d\tau = \int_{(-\omega+i\omega')/2}^{(\omega+i\omega')/2} (\zeta(\tau)\omega' i/2 - \tau\zeta(\omega' i/2))d\tau \\ - \pi i(\tau_0 + \omega/2 - i\omega'/2) - \epsilon'\pi\omega',\end{aligned} \tag{6.21}$$

with $\epsilon' = 0$ for $\omega' > \Im\tau_0 > 0$, and $\epsilon' = 1$ for $-\omega' < \Im\tau_0 < 0$. Thus

$$\begin{aligned}\delta\phi = (1/\omega')\Re \sum_{l=1}^{h} [-\pi i(i\omega'\epsilon/2 - \bar{\alpha}'_l + \omega/2 - i\omega'/2) \\ + \pi i(i\omega'\epsilon/2 - \alpha'_l + \omega/2 - i\omega'/2) - (1 - \epsilon)\pi\omega'],\end{aligned}$$

$$\delta\phi = (-1/\omega') \sum_{l=1}^{h} 2\pi\Im\alpha'_l - h(1 - \epsilon)\pi. \tag{6.22}$$

The interior boundary L_2, which has no point at infinity, cuspidal point, or point of inflection, will not cut itself if and only if $\delta\phi = +2\pi$, when the point $z = F(T)$ traces this curve, i. e. when $\epsilon = 1$. For this, it is obviously necessary and sufficient to have

$$\sum_{l=1}^{h} \Im\alpha'_l = -\omega'. \tag{6.23}$$

Now, if this condition is satisfied, $\delta\phi$ will be equal to $2\pi - h\pi$ when the point $z = F(T)$ traces the exterior curve L_1, which has h infinite branches. From this, one can easily conclude that L_1 does not cut itself.

The functions $F'(T)$ which correspond to the solutions of our problem for the special case considered in this paragraph are therefore of the form

$$F'(T) = A \prod_{l=1}^{h} (\sigma(T - \alpha_l)/\sigma(T - \beta_l)) \tag{6.24}$$

where $0 \leq \beta_1 < \beta_2 < \cdots < \beta_h < \omega$; $\alpha_1 = \alpha'_1 + i\omega'$, $\alpha_2 = \alpha'_2 \cdots, \alpha_h = \alpha'_h$, *with* $-\omega'/2 < \Im\alpha'_l < 0$, *and* $\sum_{l=1}^{h} \alpha_l - \sum_{l=1}^{h} \beta_l = 0$. *It follows from the preceding analysis that such a function satisfies the conditions* V, VI *of* § **4**. *It satisfies obviously by construction also the conditions* I, II *and* IV *of that paragraph.*

Let us show now that the zeros of any function of the form

$$F'(T) = (-2/\omega)\zeta(\omega/2) \sum_{l=1}^{h} f_l \exp(i\phi_l)\beta_l - \sum_{l=1}^{h} f_l \exp(i\phi_l)\zeta(T - \beta_l), \tag{6.25}$$

with $\sum_{l=1}^{h} f_l \exp(i\phi_l) = 0$, $0 \leq \beta_1 < \beta_2 < \cdots < \beta_h < \omega$, *and* (§ **4**, III) $f_l > 0$, $0 \leq \phi_2 - \phi_1 \leq \phi_3 - \phi_1 \leq \cdots \leq \phi_h - \phi_1 < 2\pi$ *satisfy the conditions above, where such a function has h zeros $\alpha'_1, \alpha'_2, \cdots, \alpha'_h$ in* $-\omega'/2 < \Im T < 0$, $0 \leq \Re T < \omega$ *and* $\sum_{l=1}^{h} \Im\alpha'_l = -\omega'$. We consider first the function

$$F'_0(T) = (\sigma(T + i\omega'/h - i\omega')/\sigma(T)) \prod_{l=1}^{h-1} (\sigma(T + i\omega'/h - l\omega/h)/\sigma(T - l\omega/h)), \tag{6.26}$$

$$= -\exp(-2\zeta(i\omega'/2)(T + i\omega'/h - i\omega'/2)) \prod_{l=0}^{h-1} (\sigma(T + i\omega'/h - l\omega/h)/\sigma(T - l\omega/h)),$$

which satisfies the equation

$$F'_0(T + \omega/h) = F'_0(T)\exp(2\pi i/h). \tag{6.27}$$

The coefficient A_j of the principal part of this function at the pole $(j-1)\omega/h$ is obviously the ratio of

$$-\exp(-2\zeta(i\omega'/2)((j-1)\omega/h + i\omega'/h - i\omega'/2)) \prod_{l=0}^{h-1} \sigma((j-1)\omega/h + i\omega/h - l\omega/h)$$

to

$$\sigma((j-1)\omega/h)\sigma((j-2)\omega/h)\cdots\sigma(\omega/h)\sigma(-\omega/h)\cdots\sigma((j-h)\omega/h);$$

we have

$$A_j = A_1 \exp(2(j-1)\pi i/h). \tag{6.28}$$

The function

$$F'_{0'}(T) = (2A_1/\omega)\sum_{j=1}^{h}\exp(2(j-1)\pi i/h) \times(\zeta(T-((j-1)\omega/h)\omega/2-(T-(j-1)\omega/h)\zeta(\omega/2)) \tag{6.29}$$

differs therefore from $F'_0(T)$ by a constant K. But it is easy to verify that the function $F'_{0'}(T)$ is also a solution of the linear functional equation (6.27). The constant K is therefore zero. The zeros $\alpha'_{10} = -i\omega'/h$, $\alpha'_{20} = -i\omega'/h + \omega/h, \cdots, \alpha'_{h0} = -i\omega'/h + (h-1)\omega/h$ of the function $F'_{0'}(T) = F'_0(T)$, which is of the form (6.25), are actually in the rectangle $-\omega'/2 < \Im T < 0$, $0 \leq \Re T < \omega$, and we have $\sum_{j=1}^{h}\Im\alpha'_{j0} = -\omega'$.

We consider now an arbitrary function

$$F'(T) = (-2/\omega)\sum_{j=1}^{h} f_j \exp(i\phi_j)(\zeta(T-\beta_j)\omega/2-(T-\beta_j)\zeta(\omega/2))$$

of the form (6.25). This function has h zeros $\alpha'_1, \alpha'_2, \cdots, \alpha'_h$, which are continuous functions of $\phi_1, \phi_2, \cdots, \phi_h$; $f_1, f_2, \cdots, f_h$; $\beta_1, \beta_2, \cdots, \beta_h$, and which take respectively the values $\alpha'_{10}, \alpha'_{20}, \cdots, \alpha'_{h0}$ for

$$\phi_j = \phi_{j0} = 2(j-1)\pi/h, \qquad f_j = f_{j0} = 1, \qquad \beta_j = \beta_{j0} = (j-1)\omega/h \quad (j = 1, 2, \cdots, h).$$

From

$$\sum_{j=1}^{h}\Im\alpha'_j \equiv 0 \mod \omega', \qquad \sum_{j=1}^{h}\Im\alpha'_{j0} = -\omega',$$

we deduce by continuity

$$\sum_{j=1}^{h}\Im\alpha'_j = -\omega'.$$

We will show now that the points $\alpha'_1, \alpha'_2, \cdots, \alpha'_h$, which are initially inside of the strip $-\omega'/2 < \Im T < 0$, always remain in that strip. Suppose they are not inside of that strip for $\phi_j = \phi_{j1}$, $f_j = f_{j1}$, $\beta_j = \beta_{j1}$, and let us consider a system of continuous functions $\phi_j(\lambda), f_j(\lambda), \beta_j(\lambda)$ $(0 \leq \lambda \leq 1)$ which satisfy the conditions

$$0 \leq \phi_2(\lambda) - \phi_1(\lambda) \leq \phi_3(\lambda) - \phi_1(\lambda) \leq \cdots \leq \phi_h(\lambda) - \phi_1(\lambda) < 2\pi,$$

$$f_j(\lambda) > 0, \qquad \sum_{j=1}^{h} f_j(\lambda) \exp(i\phi_j(\lambda)) = 0,$$

$$0 \leq \beta_1(\lambda) < \beta_2(\lambda) < \cdots < \beta_h(\lambda) < \omega,$$

$$\phi_j(0) = \phi_{j0}, \qquad f_j(0) = f_{j0}, \qquad \beta_j(0) = \beta_{j0},$$

$$\phi_j(1) = \phi_{j1} \qquad f_j(1) = f_{j1}, \qquad \beta_j(1) = \beta_{j1},$$

at least one of the differences $\phi_j(\lambda) - \phi_1(\lambda)$ being different from 0 and π. The numbers $\alpha'_j = \alpha'_j(\lambda)$ are then continuous functions of λ which take the values $\alpha'_{10}, \alpha'_{20}, \cdots, \alpha'_{h0}$ for $\lambda = 0$. Some of the values of $\alpha'_1(\lambda), \alpha'_2(\lambda), \cdots, \alpha'_h(\lambda)$ for $\lambda = 1$ being outside of the strip $-\omega'/ < \Im T < 0$, we can find a value λ_0 $(0 < \lambda_0 \leq 1)$ of the variable λ such that the points $\alpha'_1(\lambda)$, $\alpha'_2(\lambda), \cdots, \alpha'_h(\lambda)$ lie inside of that strip for $0 \leqq \lambda < \lambda_0$, and some of the points $\alpha'_1(\lambda_0), \alpha'_2(\lambda_0), \cdots, \alpha'_h(\lambda_0)$ become boundary points of $-\omega'/2 < \Im T < 0$, the others being still inside of it. Let us consider now the expressions

$$F(T, \lambda) = (-2/\omega) \int \sum_{j=1}^{h} f_j(\lambda) \exp(i\phi_j(\lambda))(\zeta(T - \beta_j(\lambda))\omega/2 - (T - \beta_j(\lambda))\zeta(\omega/2))dT,$$

$$F'(T, \lambda) = A \exp(-2\zeta(i\omega'/2)T) \prod_{j=1}^{h} \sigma(T - \alpha'_j(\lambda))/\sigma(T - \beta_j(\lambda)).$$

Since the points $\alpha'_j(\lambda)$ are all in $-\omega'/2 < \Im T < 0$ for $\lambda < \lambda_0$, we can say, according to what we have shown before, that $\arg F'(t + i\omega'/2, \lambda) = \phi$ changes monotonically from $\arg F'(t_0 + i\omega'/2, \lambda)$ to $\arg F'(t_0 + i\omega'/2, \lambda) + 2\pi$ if the real variable t varies from t_0 to $t_0 + \omega$, for $\lambda < \lambda_0$. By continuity we can say the same for $\lambda = \lambda_0$ with the only difference that if, for example, $\alpha'_1(\lambda_0)$ is on the line $\Im T = -\omega'/2$, then the expression $\arg F'(t + i\omega'/2, \lambda_0)$ makes a jump of magnitude π at the point $\alpha'_1(\lambda_0) + i\omega'$ of the segment $(t_0 + i\omega'/2, t_0 + i\omega'/2 + \omega)$. On the other hand, the point $F(t + i\omega'/2, \lambda_0)$ traces a closed curve when t varies from t_0 to $t_0 + \omega$. But this is only possible if this closed curve is a line segment traced twice and that would be only possible if all of the numbers $\phi_2(\lambda_0) - \phi_1(\lambda_0), \phi_3(\lambda_0) - \phi_1(\lambda_0), \cdots, \phi_h(\lambda_0) - \phi_1(\lambda_0)$ were equal to zero or π. The points $\alpha'_1(\lambda_0), \alpha'_2(\lambda_0), \cdots, \alpha'_h(\lambda_0)$ are therefore all in the strip $-\omega'/2 < \Im T \leqq 0$. It remains to show that none of them can be on the line $\Im T = 0$.

Since the points $\alpha'_j(\lambda)$ are all in the strip $-\omega'/2 < \Im T < 0$ for $\lambda < \lambda_0$, $\arg F'(t, \lambda)$ changes monotonically from $\arg F'(t_0, \lambda)$ to $\arg F'(t_0, \lambda) + 2\pi - h\pi$,

when the real variable t changes from t_0 to $t_0+\omega$; also it changes from $\phi_l - k\pi$ to $\phi_{l+1} - (k+1)\pi$ when t traces the interval between two consecutive poles $\beta_l(\lambda) + N\omega$ and $\beta_{l+1} + M\omega$ of $F'(T,\lambda)$. By continuity one can say the same for $\lambda = \lambda_0$ with the only difference that if, for example, $\alpha_1(\lambda_0)$ is in the interval $(\beta_l(\lambda_0) + N\omega, \beta_{l+1}(\lambda_0) + M\omega)$, then the expression $\arg F'(t,\lambda_0)$ will make a jump of magnitude $-\pi$ at the point $\alpha_1(\lambda_0)$. We shall have therefore

$$(\phi_{l+1}(\lambda_0) - (k+1)\pi) - (\phi_l(\lambda_0) - k\pi) < -\pi,$$

or $\phi_{l+1}(\lambda_0) - \phi_l(\lambda_0) < 0$, which contradicts our hypothesis $\phi_{l+1}(\lambda_0) > \phi_l(\lambda_0)$.

We can now summarize the results of this paragraph as follows.

One obtains a solution of our problem, for $m=2$ and σ_x, σ_y, τ finite everywhere in C, by putting:

$$z = F(T) = (-2/\omega)\sum_{j=1}^{h} f_j \exp(i\phi_j)\int(\zeta(T-\beta_j)\omega/2 - (T-\beta_j)\zeta(\omega/2))dT,$$

$$= -\sum_{j=1}^{h} f_j \exp(i\phi_j)\log\theta_1((T-\beta_j)/\omega),$$

with $0 \leq \phi_2 - \phi_1 \leq \phi_3 - \phi_1 \leq \cdots \leq \phi_h - \phi_1 < 2\pi$, $0 \leq \beta_1 < \beta_2 < \beta_3 < \cdots \leq \beta_h < \omega$, $f_j > 0$, $\sum_{j=1}^{h} f_j \exp(i\phi_j) = 0$; *and* $H = \underline{F}'(T)/F'(T)$

$$= \sum_{j=1}^{h} f_j \exp(-i\phi_j)(\zeta(T-\beta_j)\omega/2 - (T-\beta_j)\zeta(\omega/2))$$

$$\times \sum_{j=1}^{h} f_j \exp(i\phi_j)(\zeta(T-\beta_j)\omega/2 - (T-\beta_j)\zeta(\omega/2))$$

$$\sigma_x = 2\alpha(1+\Re H(T)), \qquad \sigma_y = 2\alpha(1-\Re H(T)), \qquad \tau = -2\alpha\Im H(T).$$

The domain C is then the image of the rectangle $\omega'/2 > \Im T > 0$, $0 \leq \Re T < \omega$, in the z plane, and every solution of that problem can be obtained in this way.

Appendix.

Let ω, ω' be real and positive; then the following inequalities hold for the Weierstrass ζ function corresponding to the periods ω and $i\omega'$:

$$\Re(\zeta(T)i\omega'/2 - T\zeta(i\omega'/2)) \geq 0 \quad \text{for } n\omega' \leq \Im T \leq \omega'/2 + n\omega',$$

$$\Re(\zeta(T)i\omega'/2 - T\zeta(i\omega'/2)) \leq 0 \quad \text{for } -\omega'/2 + n\omega' \leq \Im T \leq n\omega',$$

and equality holds only on the boundaries of the strips.

Proof. Because of the relations

$$\zeta(T+\omega)i\omega'/2 - (T+\omega)\zeta(i\omega'/2)$$
$$= \zeta(T)i\omega'/2 - T\zeta(i\omega'/2) + i\omega'\zeta(\omega/2) - \omega\zeta(i\omega'/2),$$
$$= \zeta(T)i\omega'/2 - T\zeta(i\omega'/2) + \pi i,$$
$$\zeta(T+i\omega')i\omega'/2 - (T+i\omega')\zeta(i\omega'/2)$$
$$= \zeta(T)i\omega'/2 - T\zeta(i\omega'/2) + i\omega'\zeta(i\omega'/2) - i\omega'\zeta(i\omega'/2)$$
$$= \zeta(T)i\omega'/2 - T\zeta(i\omega'/2),$$

it is sufficient to prove the inequalities for $n = 0$ and $-\omega/2 < \Re T \leq \omega/2$. According to the relation of Legendre, the function

$$g(T) = \zeta(T)i\omega'/2 - T\zeta(i\omega'/2)$$

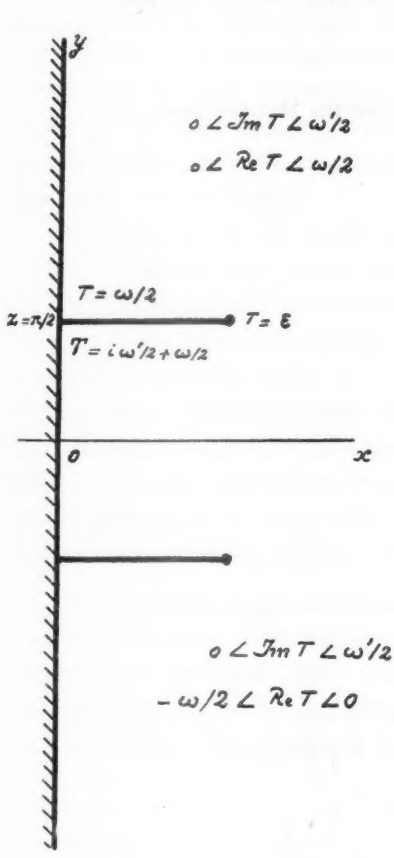

FIG. 4.

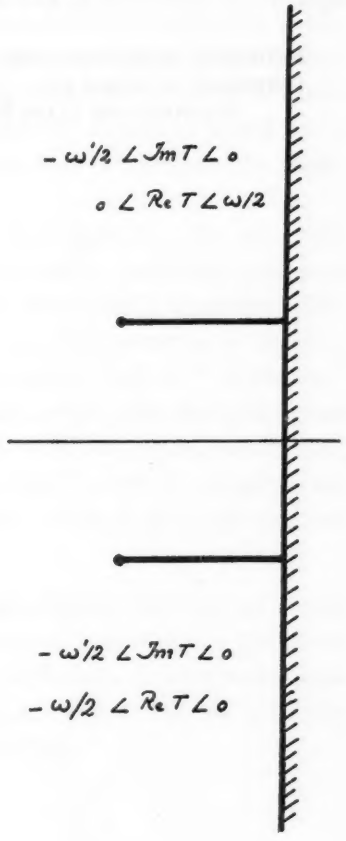

FIG. 5.

takes the same value $i\pi/2$ for $T = \omega/2$ and for $T = (\omega + i\omega')/2$. Its derivative

$$g'(T) = -(\mathfrak{P}(T)i\omega'/2 + \zeta(i\omega'/2))$$

is purely imaginary on the segment $(\omega/2, (\omega + i\omega')/2)$. This derivative has therefore a zero $T = \epsilon$ on this segment. The other zero of $g'(T)$ in the period rectangle $(-i\omega'/2, -i\omega'/2 + \omega, i\omega'/2 + \omega, i\omega'/2)$ is then $T = \bar{\epsilon}$. The function $z = g(T)$ therefore maps the rectangle $(0, \omega/2, (\omega + i\omega')/2, i\omega'/2)$ conformally on the domain shown in fig. 4. By symmetry with respect to the segment $(i\omega'/2, 0)$ we see that the rectangle $(-\omega/2, \omega/2, (\omega + i\omega')/2, -\omega/2 + i\omega'/2)$ is mapped on the right side of the imaginary axis of the z plane, while the rectangle $(-\omega/2, \omega/2, \omega/2 - i\omega'/2, -\omega/2 - i\omega'/2)$ is mapped on the left side of this axis (symmetrically with respect to the segment $(-\omega/2, \omega/2)$), and our inequalities then become obvious.

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A PERIODIC ALGORITHM FOR CUBIC FORMS.*

By HARVEY COHN.

1. Introduction. In a very famous memoir in 1896, Minkowski [5] set up a continued fraction type of algorithm for the product of three real linear homogeneous forms in three integral variables. By the precedent set in the ordinary theory of binary quadratic forms, such an algorithm has to supply a definition of *reduced form* with several properties: First of all every form should be equivalent under unimodular change of variables to one or more reduced forms. Furthermore a concept of *neighboring form* should be present so that the reduced forms will arrange themselves into chains. Finally when the three linear forms are conjugates, each spanning a (totally real) *cubic module,* the number of distinct reduced forms should be *finite.* Probably the most immediate application of such an algorithm would be the determination of units of a totally real cubic field as automorphs within the finite chain of forms.

The purpose of this work is to present a new algorithm for such cubic forms, combining some of the features of an ordinary (modified) continued fraction algorithm with some of the features of Minkowski's algorithm. This new algorithm follows the former one to the extent that reduction is based on a visually convenient array of + and — signs rather than on "admissible" bodies; and it follows the latter algorithm to the extent that the proof given here for the reduction theorem uses Minkowski's work as a convenient starting point. Once the theoretical aspects are established, it will be seen that the actual calculations in cubic fields are very simple, although the units obtained there are far from thoroughly investigated.

2. Modified simple continued fraction algorithm. The familiar simple continued fraction (for binary quadratic forms) of course fulfils the above requirements *par excellence.* We present a modification to emphasize some curious resemblances to the definition of reduced forms that we presently give for cubics. The quadratic form may be written

$$\Phi(n, m) = x_1 x_2, \tag{2}$$

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where $x_i = n - \theta_i m$, $(n, m$ integers), $i = 1, 2$ and θ_i are conjugate real quadratics. The condition for reduction (as stated by Gauss) is that θ_1 and $-1/\theta_2$ be each greater than 1. If we consider the lattice in the (x_1, x_2)-plane we find this is *almost the condition that* V_1, V_2, V_3, *the* (x_1, x_2)*-vectors corresponding to* $(n, m) = (0, 1)$, $(1, 0)$, $(-1, -1)$ *respectively, do not all three lie in one pair of opposite quadrants of the* (x_1, x_2)*-plane.* Here note that V_1, V_2, V_3 have the property that any two of the three are basis vectors of the lattice, and $V_1 + V_2 + V_3 = 0$. The idea of using not two but *three* such vectors (but in another formulation), goes back to Selling [1]. The concept of neighbor in the simple continued fraction is (again) *almost the condition that two of the three vectors* V_1, V_2, V_3 *be common to the two forms in question.* The resemblance of this modified continued fraction algorithm to the one presented in this paper will become clear as matters develop.

3. Minkowskian basis. We now use as a point of departure the existence of a Minkowskian basis, an immediate consequence of Minkowski's algorithm.

We recall that a lattice is said to be *admissible* with respect to a convex body K (or vice versa) if no point of the lattice except the origin is interior to K. The convex body which we use is the rectangular parallelopiped:

$$P(\xi_1, \xi_2, \xi_3) : |x_i| \leq \xi_i. \tag{3.1}$$

The lattice we use can be written as

$$x_i = b_{i1}m_1 + b_{i2}m_2 + b_{i3}m_3, \qquad m_j \text{ integral}, i, j = 1, 2, 3. \tag{3.2}$$

Its determinant D is assumed $\neq 0$, and we henceforth make the simplifying assumption that *none of the components* x_1, x_2, *or* x_3 *shall vanish for any vector of the lattice except the origin.* This fits in conveniently with the intended application, where the triple (b_{i1}, b_{i2}, b_{i3}) spans the totally real cubic field K_i (where the index i denotes the conjugate).

We consider an ordered triple of lattice vectors (A_1, A_2, A_3), where $A_i = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$. We call such a triple *Minkowskian* if $P(x_1^{(1)}, x_2^{(2)}, x_3^{(3)})$ is admissible. Such triples exist under very broad assumptions ([2]) and *from Minkowski's algorithm* ([5]) *such a triple is either co-planar or a lattice basis.* (The proof consists essentially, [4], of eliminating the possibility that the triple generate a sub-lattice of index 2, and this can be done more simply by the methods of **5** below).

Minkowski's algorithm also supplies a process of forming neighbors which we need not discuss in detail except for the following points of imme-

diate concern: The process [3] of "lowering the faces" shows that every Minkowskian triple has exactly *three* neighbors. The process, in addition, involves a "visualization" of the lattice with a view to admissibility. (The neighbor process proposed in **10** below will be less simple in the first respect but much more simple in the second.)

Minkowski's neighbor process is also such that a co-planar Minkowskian triple (fortunately) cannot have co-planar neighbors ([3]); hence we have the

THEOREM. *Every lattice has a Minkowskian triple that serves as a basis.*

We call this a *Minkowskian basis* and proceed from there.

Reduction Theorem.

4. Definition of reduced form. We start off by considering a 3×4 *matrix* ϕ written as four column vectors P, Q, R, S with three x_i components p_i, q_i, r_i, s_i:

$$\phi: \quad \begin{matrix} & P & Q & R & S \\ (x_1:) & p_1 & q_1 & r_1 & s_1 \\ (x_2:) & p_2 & q_2 & r_2 & s_2 \\ (x_3:) & p_3 & q_3 & r_3 & s_3 \end{matrix}. \tag{4.1}$$

Such a matrix is said to be reduced if

$$O = P + Q + R + S \tag{4.21}$$

and if

$$\pm \operatorname{sgn} \phi = \begin{Bmatrix} + & - & - & + \\ - & + & - & + \\ - & - & + & + \end{Bmatrix}; \tag{4.22}$$

(i. e., if the components of ϕ have either the exact or reverse pattern of $+$ signs and $-$ signs of the last matrix). Such a matrix is written out in **11** (below).

From the definition it is clear that no three of the column vectors P, Q, R, S are co-planar (for, if so, there would be a common perpendicular to all four of them in contradiction to the fact that these four vectors and their negatives lies one in each octant).

We next define classes of *proportional* matrices, by saying ϕ and ϕ' are proportional, i. e.

$$\phi \sim \phi', \tag{4.31}$$

if the three (x_i) rows of each are proportional, $p_i/p'_i = q_i/q'_i = r_i/r'_i = s_i/s'_i = w_i(\neq 0)$. We say that ϕ and ϕ' are *strictly proportional,* or

$$\phi \approx \phi', \tag{4.32}$$

if $\phi \sim \phi'$ and, in addition, $w_1 w_2 w_3 = 1$.

Clearly if $\phi \sim \phi'$ and ϕ' is reduced, then ϕ^λ is reduced, where λ is a permutation in the Klein four-group on (P, Q, R, S). (If all constants, w_i, are positive, then λ is the identity.)

We next consider *degenerate ternary forms,* i. e.

$$\Phi = \sum_{ikl}^{3} = a_{ikl} m_i m_k m_l, \tag{4.41}$$

where

(4.42)

$$\Phi = x_1 x_2 x_3; \qquad x_i = b_{i1} m_1 + b_{i2} m_2 + b_{i3} m_3, \qquad 1 \leq i, j \leq 3;$$

$$B_j = (b_{1j}, b_{2j}, b_{3j}).$$

We call such a form *reduced,* or we say the three column vectors B_1, B_2, B_3 *are a reduced basis,* if these three vectors, taken in some order, come from the column vectors P, Q, R, S of a reduced 3×4 matrix. This definition is immediately invariant under permutation of variables m_1, m_2, m_3. This definition is such that given a form Φ, the question of whether or not it is reducible does not depend on how the factors x_i are identified. In fact if the factors x_i are transformed by the appropriate group: $x_1 \to c_1 x_i$, $x_2 \to c_2 x_j$; $x_3 \to c_3 x_k$, $i \neq j \neq k$, $(c_1 c_2 c_3 = 1)$, the corresponding 3×4 matrix obeys $\phi \to \phi'$, where

$$\phi' \approx \phi^\sigma, \tag{4.5}$$

and σ denotes a permutation from the alternating group on (P, Q, R, S).

We could finally discuss a *degenerate quaternary form* given by formulas similar to (4.41) and (4.42) except that the index j would go from 1 to 4 and there also would be 4 values of m_j. Then a *reduced* form would be one in which the B_1, B_2, B_3, B_4 were, in some order, the P, Q, R, S of a reduced 3×4 matrix. Thus any reduced degenerate *ternary* form Φ results from a reduced degenerate *quaternary* form in which the vanishing of a single m_i leaves the form equal to Φ (in the three remaining variables). The quaternary forms are such that each *strict* proportionality class of ϕ determines the same form (to within permutation of the m_i) together with exactly all the ϕ^σ described in (4.5). The table in **13** (below) will provide instances where

4 or 12 such ϕ determine the same quaternary form. Such properties necessarily extend to the ternary forms.

5. Proof that every degenerate ternary form is equivalent to a reduced form.

THEOREM. *If A_1, A_2, A_3 are a Minkowskian basis, then $\pm A_1$, $\pm A_2$, $\pm A_3$, with an appropriate choice of signs, is a reduced basis.*

Remark. It seems that some significance was attached to the 3×3 array of signs of $\pm A_i$ by Minkowski [5], but the role of a fourth vector was not considered.

Proof. We can without loss of generality multiply the A_i by constants and reflect the coordinate planes so as to have

$$a_{ii} = 1, \quad |a_{ij}| < 1 \qquad \text{if } i \neq j, a_{21} > 0, a_{31} > 0, \tag{5.11}$$

$$A_1 = (1, a_{12}, a_{13}), \quad A_2 = (a_{21}, 1, a_{23}), \quad A_3 = (a_{31}, a_{32}, 1). \tag{5.12}$$

Now the hypothesis that A_1, A_2, A_3 is a Minkowskian basis requires, among other things, that the 20 vectors $\pm A_i \pm A_j$ $(i \neq j)$, $\pm A_1 \pm A_2 \pm A_3$ lie on or outside the cube $|x_i| < 1$. So from here on, we *weaken the hypothesis,* (see [4]) so that the a_{ij} satisfy the conditions in (5.11) together with merely

$$\begin{aligned} &\max_k |\pm a_{ik} \pm a_{jk}| > 1 \qquad (i \neq j), \\ &\max_k |\pm a_{1k} \pm a_{2k} \pm a_{3k}| > 1, \text{ for each choice of } \pm. \end{aligned} \tag{5.2}$$

We now assert that the pattern of signs of the four vectors

$$\pm A_1, \pm A_2, \pm A_3, \mp A_1 \mp A_2 \mp A_3, \tag{5.3}$$

does in some order accord with the system (4.22), if the $\pm$ signs are chosen properly for each $\pm A_i$.

Next we visualize the Π_1: (a_{21}, a_{31})-plane, the Π_2: (a_{12}, a_{32})-plane, and the Π_3: (a_{13}, a_{23})-plane, written more generally as (ξ, η)-planes. We see that the regions determined by the inequalities (5.11) and (5.2) are bounded by lines of the types: $\pm\xi = 0, 1$; $\pm\eta = 0, 1$; $\pm\xi \pm \eta = 0, 1$. Thus in the $\Pi_1 \times \Pi_2 \times \Pi_3$ space, $4 \times 16 \times 16 = 1024$ regions with isoscles right-triangular projections are determined. In fact, considering the sixteen Π_2- and the sixteen Π_3-projection triangles, we see that each one contains in its interior exactly one point of one of the 16 types $(\pm\frac{1}{2}, \pm\frac{1}{4})$, $(\pm\frac{1}{2}, \pm\frac{3}{4})$, $(\pm\frac{1}{4}, \pm\frac{1}{2})$, $(\pm\frac{3}{4}, \pm\frac{1}{2})$, while the Π_1-projection triangle contains only of the four types

formed using only + signs, $(a_{k1} > 0)$. In each of these 1024 (open) regions the inequalities in (5.11) are always true, while the inequalities in (5.2) are *each* always true or always false. In the 184 cases where (5.11) and (5.2) are *always* true it is easy to choose the vectors in (5.3) to have the proper array of signs. The handling of all 1024 cases can be easily accomplished in a few hours by using $4 + 16 + 16$ strips of paper with the x_1-, x_2-, and x_3-components of the 26 vectors $\pm A_i$, $\pm A_i \pm A_j$, $\pm A_1 \pm A_2 \pm A_3$, $(i \neq j)$.

6. Alternative form of the reduction theorem. We have shown that every basis is equivalent under change of basis to a reduced basis. It is no loss of generality to assume that the change of basis involves a matrix of determinant $+1$ (as *all* basis vectors can be multiplied by -1 without affecting the definition of reduction). Likewise we may assume by (4.21) that B_1, B_2, B_3 are S, P, Q respectively. Finally we divide each x_i by a constant so as to put the lattice in inhomogeneous form, whence $b_{1i} = 1$. Then calling $b_{2i} = \xi_i$, $b_{3i} = \eta_i$ we obtain the following result:

Let three non-collinear points (ξ_i, η_i) of the projective $(\xi\eta)$-plane be given in some fixed order. It is possible to find an integral unimodular transformation such that the points (ξ_i, η_i) simultaneously go into the points (ξ'_i, η'_i),

$$(6.1) \qquad \xi'_i = \frac{a_1 + b_1\xi_i + c_1\eta_i}{a_3 + b_3\xi_i + c_3\eta_i}, \qquad \eta'_i = \frac{a_2 + b_2\xi_i + c_2\eta_i}{a_3 + b_3\xi_i + c_3\eta_i}, \qquad \det(a_1b_2c_3) = +1$$

lying in the following three regions of the projective plane respectively:

$$(6.2) \qquad \begin{aligned} R_1&: \ \xi_1 \geq 0, \eta_1 \leqq 0, \qquad 1 + \xi_1 + \eta_1 \geq 0; \\ R_2&: \ \xi_2 \leq 0, \eta_2 \geqq 0, \qquad 1 + \xi_2 + \eta_2 \geqq 0; \\ R_3&: \ \xi_3 \leq 0, \eta_3 \leqq 0, \qquad 1 + \xi_3 + \eta_3 \leq 0. \end{aligned}$$

Note that in order to cope with our previous lattice restriction (that no lattice vector have a zero component except the origin), we must extend the regions to the projective plane, thus permitting division by zero.

7. Cyclic basis. A very important type of lattice is one with a cyclic basis,

$$(7.1) \qquad b_{ij} = \alpha_k, \text{ where } k \equiv i + j \pmod 3 \qquad (k = 1, 2, 3).$$

In this case the ternary form Φ, as a function of the m_i, is invariant under cyclic permutations, and conversely. Here the basis B_j, $(j = 1, 2, 3)$, (in the notation of **4**) is reduced if and only if, of the four quantities α_1, α_2, α_3, $-(\alpha_1 + \alpha_2 + \alpha_3)$, two are positive and two are negative,

$$(\text{or } \alpha_1\alpha_2\alpha_3(\alpha_1 + \alpha_2 + \alpha_3) < 0).$$

We now single out the "hard case," where, under a change of indices and (possibly) a change in sign of all α_i, we have the inequalities

$$\alpha_1 + \alpha_2 > \alpha_3 > \alpha_2 > \alpha_1 > 0. \tag{7.2}$$

Then, except for the hard case just cited, the cyclic basis will be finitely close to a reduced basis in the sense that the vectors P, Q, R, S for the reduced basis can be chosen from among the 26 vectors $\pm B_i$, $\pm B_i \pm B_j$, $\pm B_1 \pm B_2 \pm B_3$, $(i \neq j)$.

The above statement can be easily verified. We further note that the hard case must be separated since the three numbers α_1, α_2, α_3 can otherwise be arbitrarily close to one another and still satisfy the conditions (7.2). Therefore, even if the set of 26 vectors above were extended to any finite set, however large, of the type $k_jB_1 + l_jB_2 + m_jB_3$, it would still be possible to choose the α_i so that of the four desired basis vectors P, Q, R, S, the one (S) which is in the all + (or all —) octant would be the one for which $k_j + l_j + m_j \neq 0$, while the other three vectors would be such that $k_j + l_j + m_j = 0$, contradicting condition (4.21).

Fortunately, in many applications, the hard case simply does not occur. For instance, for later reference, take

$$\alpha_k = \sum \exp(2\pi i m/p), \qquad (k = 1, 2, 3), \tag{7.3}$$

where p ($\equiv 1 \bmod 3$) is a prime and m through a complete set of residues of cubic character indicated by k. Here $\alpha_1 + \alpha_2 + \alpha_3 = -1$, and therefore if the α_i were all of one sign (as in (7.2)), we should have $|\alpha_1\alpha_2\alpha_3| \leqq 1/27$ in contradiction to the fact that α is an algebraic integer. Hence the lattice given by (7.1) and (7.3), (determined by contraction of a cyclotomic field), has a reduced basis very close to the cyclic basis described here.

Finiteness.

8. Norms and discriminants. We now consider the case where the x_1, x_2, x_3 components in (4.42) are the conjugates of three numbers spanning a (totally real) cubic field. We shall show that although the neighbor process, for instance, produces a chain of 3×4 matrices (see **12** below), they are strictly proportional to only a finite number of classes. Thus the ternary and quaternary forms take on only a finite number of values.

As a preliminary step, multiply through the matrices so as to make all of the p_i, q_i, r_i, s_i algebraic integers. We let Δ represent the (common) absolute value of the determinants formed by omitting any one of the four

columns. Here Δ^2 is an integer which represents the field discriminant multiplied by j^2, where j represents the index of the sub-lattice (or module) of integers in the ring of all integers of the field. Of course Δ is invariant under change of basis.

In the usual notation, where sub-scripts denote conjugates, we denote the norm and discriminant of the vector $A = (a_1, a_2, a_3)$ by means of

$$(8.1)\quad N(A) = a_1a_2a_3; \qquad (8.2)\quad d(A) = [(a_1 - a_2)(a_2 - a_3)(a_3 - a)]^2.$$

Clearly $d(A) = 0$ only if $a_1 = a_2 = a_3 =$ rational number, while $N(A) = 0$ only if $a_1 = a_2 = a_3 = 0$. Furthermore when A is an algebraic integer, N and d are natural integers.

We shall next show that when Δ is bounded (and > 0) the norms of at least *three* of the four algebraic integers P, Q, R, S are bounded ($< (\Delta/3)^3$) while the discriminants of *all twelve* fractions of the type $P/Q = (p_i/q_i)$ are likewise restricted to a finite set of rational fractions (whose numerators go from 1 to $[(\Delta/3)^6]$ and whose denominators go from 1 to $[(\Delta/3)^{18}]$).

The above conclusions follow from the expansion of Δ in (4.1) along (say) columns R and S using, each time, the minors of P and Q.

$$(8.3)\quad \begin{aligned} -\Delta &= (p_1q_1 - q_1p_2)r_3 + (p_3q_1 - q_3p_1)r_2 + (p_2q_3 - q_2p_3)r_1, \\ +\Delta &= (p_1q_2 - q_1p_2)s_3 + (p_3q_1 - q_3p_1)s_2 + (p_2q_3 - q_2p_3)s_1. \end{aligned}$$

Now take either the upper or the lower lines of (8.3) according as $(p_1q_2 - q_1p_2)$ is negative or positive. (Note that $p_iq_j - p_jq_i \neq 0$ for $i \neq j$ by the rational independence of P and Q.) In (say) the first case, reference to system (4.22) reveals that $-\Delta$ is the sum of three negative quantities, abbreviated $t_1 + t_2 + t_3$, whence easily,

$$(8.4)\qquad (\Delta/3)^3 \geqq |\, t_1t_2t_3 \,| = |\, N(R) \,| \cdot |\, G(P, Q) \,|,$$

where $G^2(P, Q) = (p_1q_2 - q_1p_2)^2(p_3q_1 - q_3p_1)^2(p_2q_3 - q_2p_3)^2$. Since $G^2(P, Q)$ is a rational integer, it follows that one column, R, of the chosen pair R, S, has bounded norm. By repeating the argument we see that some three of the column vectors (say) P, Q, R have bounded norm (as indicated above). Likewise, regardless of whether R or S has the bounded norm,

$$(8.5)\qquad G^2(P, Q) = N^4(Q)d(P/Q) = N^4(P)d(Q/P);$$

hence any of the twelve fractions of the type P/Q has, from (8.4), a discriminant restricted to the set of values indicated above.

9. Completion of finiteness proof. Using the previous information let us suppose that fixed values of $N(P/R)$, $N(Q/R)$, $d(P/R)$, $d(Q/R)$, $d(P/Q)$,

$d(S/R)$ are given as rational numbers $g_1, \cdots, g_6$ respectively, chosen from a finite set. We next see that then P/R and Q/R have only a finite set of values.

To see this we write $P/R = (u_i)$, $Q/R = (v_i)$, $P/Q = (u_i/v_i)$, $S/R = (-1 - u_i - v_i)$ by virtue of (4.21). Thus we find that

$$
\begin{aligned}
&u_1u_2u_3 = g_1, \quad v_1v_2v_3 = g_2,\\
&(u_1 - u_2)(u_2 - u_3)(u_3 - u_1) = \pm\sqrt{g_3},\\
&(v_1 - v_2)(v_2 - v_3)(v_3 - v_1) = \pm\sqrt{g_4},\\
&(u_1/v_1 - u_2/v_2)(u_2/v_2 - u_3/v_3)(u_3/v_3 - u_1/v_1) = \pm\sqrt{g_5},\\
&(u_1 + v_1 - u_2 - v_2)(u_2 + v_2 - u_3 - v_3)(u_3 + v_3 - u_1 - v_1) = \pm\sqrt{g_6}.
\end{aligned}
\tag{9.1}
$$

This constitutes six equations in the six variables u_i, v_i. There are either no more than $3^5 \cdot 6 \cdot 2^4$ solutions or else infinitely many. The latter possibility is somewhat tedious to exclude, but if there were infinitely many solutions for some g_i (all $\neq 0$), it would be possible to parametrize a one dimensional manifold in terms of $t = u_1/u_3$ as (say) $t \to \infty$. By trying to calculate from (9.1) the order of magnitudes in t of the u_i and v_i, we see that this possibility leads to a contradiction.

Now knowing that the values of

$$p_i/r_i, \ q_i/r_i \text{ and } s_i/r_i \qquad (= -1 - p_i/r_i - q_i/r_i)$$

are finite in number we can complete the proof. For then the 3×4 matrices ϕ belong to only a finite number of proportionality classes. But since the number of values of $p_1p_2p_3$ $(= N(P))$ is limited, there must also be a finite number of strict proportionality classes. Finally (see **4**) we have a bound on the number of forms Φ.

Neighbors and Chains.

10. Definition of neighbors. Our definition of neighbors will be given as a definition of neighboring *reduced* 3×4 matrices ϕ, but it can be carried over to the corresponding ternary or quaternary forms Φ. Since the order of the column vectors P, Q, R, S is fixed by the array of signs, we need only supply the "parabolic" generator of the chain. In fact we take the simplest ones consistent with condition (4.21), namely those of the type

$$\phi \to \phi' \quad (P \to P, \ \ Q \to Q + P, \ \ R \to R, \ \ S \to S - P) \tag{10.1}$$

which are written for convenience as

$$(10.2) \qquad \phi(Q.P.S)\phi' \quad \text{or} \quad \phi'(S.P.Q)\phi.$$

Of the 24 matrices ϕ' thus formed, we call the *reduced* matrices *neighbors* of ϕ.

This process creates 3, 4, 5, or 6 neighbors, depending on the relative magnitude of the components in the matrix ϕ. Furthermore, (positively) proportional reduced matrices have the same neighbors. To calculate the neighbors is very easy in that it requires not even a mental picture of the lattice as did the Minkowskian algorithm. The automatic process will now be given in terms of an illustration.

11. Illustrative calculation. For instance consider the lattice with reduced basis, (compare 7), $b_{ij} = 2\cos 2\pi ij/7$. We obtain the array of (approximate) numerical values:

$$\begin{array}{cc} & \begin{array}{cccc} P & Q & R & S \end{array} \\ \phi_0: \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \left\{\begin{array}{cccc} 1.25 & -.45 & -1.80 & 1 \\ -1.80 & 1.25 & -.45 & 1 \\ -.45 & -1.80 & -1.25 & 1\,. \end{array}\right. \end{array}$$

Looking at the x_1 row we find the elements of *like* sign compare as follows in numerical magnitude:

$$(11.1) \qquad |r_1| > |q_1|, \qquad |p_1| > |s_1|,$$

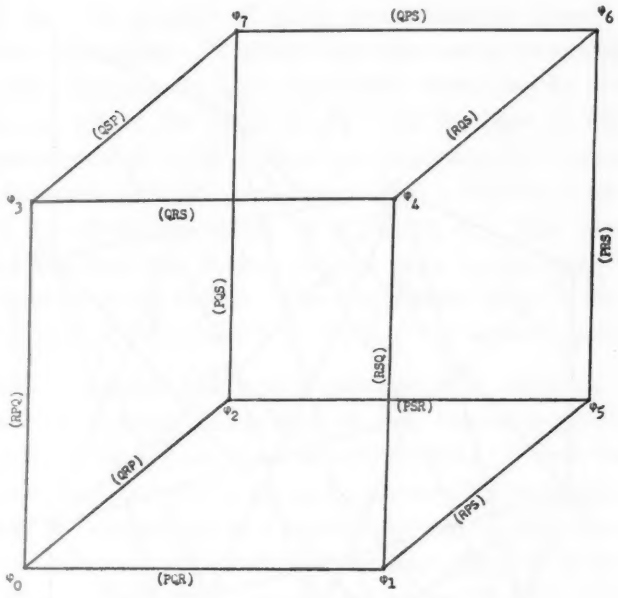
written as $(\ .R.Q)$, $(\ .P.S)$. The elements of unlike sign compare as follows in numerical magnitude:

$$(11.2) \qquad |p_1| < |r_1|, \quad |q_1| < |p_1|, \quad |q_1| < |s_1|, \quad |s_1| < |r_1|,$$

written as $(P.R.\ )$, $(Q.P.\ )$, $(Q.S.\ )$, $(S.R.\ )$ respectively.

Looking at the x_2 row we obtain six more entries, $(\ .P.R.)$, $(\ .Q.S.)$; $(Q.P.\ )$, $(S.P.\ )$, $(R.Q.\ )$, $(R.S.\ )$; and looking at the x_3 row we obtain six final entries, $(\ .Q.P.)$, $(\ .R.S)$; $(P.R.\ )$, $(P.S.\ )$, $(R.Q.\ )$, $(S.Q.\ )$.

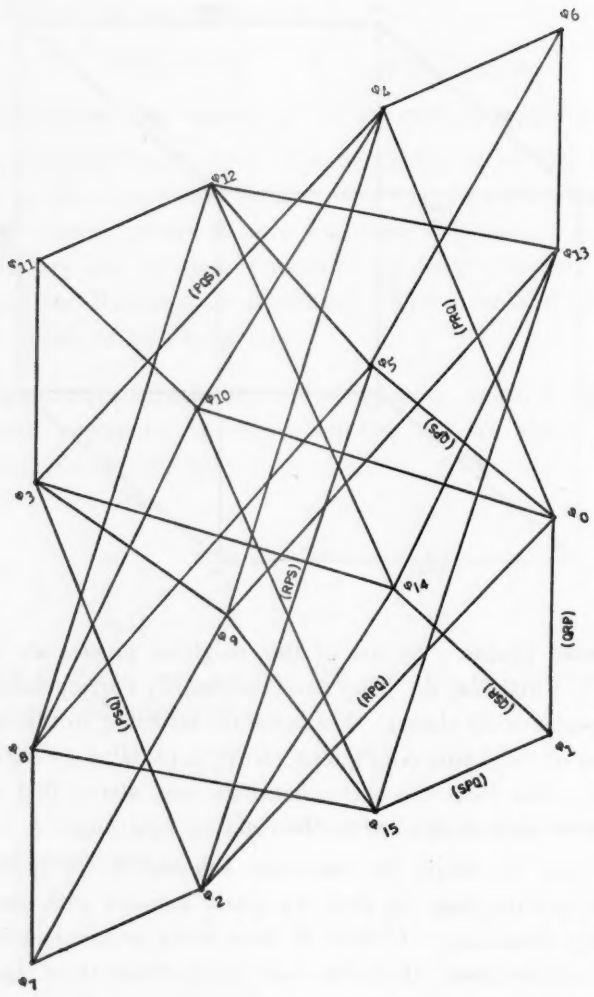
We now have 18 entries (possibly repetitive), and the rule is simply that the neighbors are those remaining when the triples of the above 18 types are *eliminated*. Thus going back (11.1) we see the entry $(\ .R.Q)$ *disposes of* $(P.R.Q)$ and $(S.R.Q)$ as neighbors, while in (11.2), the entry $(P.R.\ )$ *disposes* of $(P.R.Q)$ and $(P.R.S)$, etc. It is seen that only three neighbors for ϕ_0 are left, namely $\phi_0(P.Q.R)\phi_1$, $\phi_0(Q.R.P)\phi_2$, $\phi_0(R.P.Q)\phi_3$, which occur in the lower left hand corner of figure 1 below.



12. Closed chains. By use of this neighbor process we obtain a succession of 3×4 matrices ϕ. They must necessarily correspond to a finite set of strict proportionality classes. For instance, referring to the case we obtain the succession of neighbors $\phi_0(PQR)\phi_1(RPS)\phi_5(RSP)\phi_2(PRQ)\phi_8$; it is not hard to guess even from the approximations used above, that $\phi_8 \simeq \phi_0$, and indeed it can be seen to check with the defining equations.

In this way we obtain by succession sub-sets of the matrices in each strict proportionality class (in fact, we obtain sub-sets with *positive* relative proportionality constants). Certain of these strict proportionality classes are neighbors to one another. Hence we may always draw these classes as points connected by lines indicating neighbors. This is done in figures 1 and 2 (q. v., also see tables below).

13. Further calculations. We submit now, five calculations of closed chains, summarized in the accompanying table. The modules taken are some of the integral modules with lowest discriminant. The case $\Delta^2 = 196$ is the sub-module of index 2 of $\Delta^2 = 49$, while $\Delta^2 = 148$ is, of course, non-abelian. The (x_i) conjugates are taken in an order consistent with the sign matrix (4. 22).



Δ^2	Defining Equation	P	Q	R	S	Number of prop. classes
49	$\rho^3+\rho^2-2\rho-1=0$	ρ	$1-\rho-\rho^2$	$-2+\rho^2$	1	$2\times 4 = 8$
81	$\rho^3-3\rho-1=0$	ρ	$-2-\rho+\rho^2$	$1-\rho^2$	1	$1\times 4+1\times 12=16$
148	$\rho^3-4\rho+2=0$	$-3+\rho+\rho^2$	$2-\rho^2$	$-\rho^2$	1	$7\times 4 = 28$
169	$\rho^3+\rho^2-4\rho+1=0$	$-3+2\rho+\rho^2$	$3-\rho-\rho^2$	$-1-\rho$	1	$1\times 4+2\times 12=28$
196	$\rho^3+\rho^2-2\rho-1=0$	ρ	$-\rho-\rho^2$	$-2+\rho^2$	2	$1\times 4+6\times 12=76$

In each case the number of strict proportionality classes (see **4**) of 3×4 matrices is computed. Naturally more than one of these proportionality classes may correspond to the same quaternary form (see **4**) under permutation of the m_i, (or of the P, Q, R, S). On the basis of simple group-theoretical considerations, if more than one proportionality class corresponds to a given quaternary form the correlation is g to 1 where g is the order of a sub-group of the alternating group on P, Q, R, S. This correspondence (of classes of matrices into forms) for the cases in the table is by fours (Klein four-group) or by twelves. For non-abelian cases, if the correspondence is not unique, it is necessarily by fours, as can be easily proved.

14. Units. Consider two 3×4 matrices in a chain or, even more generally, two such matrices having three of their four column vectors respectively equivalent under a unimodular transformation. If such two matrices are *proportional*, they are *strictly proportional*, since the constants of proportionality, being the eigenvalues of a unimodular matrix, have the product 1. Since the eigenvalues are also conjugate algebraic integers, in the conjugate cubic fields, they are either all 1 or are conjugates of a non-trivial unit of the field. From the rational independence of the column vectors, the eigenvalues are all 1 only when the two 3×4 matrices are identical. In fact in the example ($\Delta^2 = 81$) of fig. 2, there are even closed circuits such as $\phi_0(Q.P.R)\phi_2(R.P.S)\phi_5(S.P.Q)\phi_0$ which produce the trivial unit. (The circuit $\phi_0(Q.P.R)\phi_2(P.S.Q)\phi_3(P.Q.S)\phi_4(Q.R.P)\phi_0$ incidentally produces a non-trivial unit). In each of the five cases treated here, independent (non-trivial) units can be observed. It is clear that as we take an increasing large number of different fields the *a priori* necessity of obtaining different units will increase the complexity of the non-trivial circuits, but no further speculation on this will be attempted at present.

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ON HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS.*

By PHILIP HARTMAN and AUREL WINTNER.

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1. Introduction. In the standard existence and uniqueness theorems for the hyperbolic differential equation

$$z_{xy} = f(x, y, z, p, q), \tag{1}$$

where $p = z_x$, $q = z_y$, it is assumed that f is continuous in (x, y, z, p, q) and satisfies a uniform Lipschitz condition with respect to (z, p, q). It will be shown (Section **2**) that the assumption of a Lipschitz condition with respect to z can be omitted in these existence theorems, though not in the uniqueness theorems. On the other hand, it will be shown in Section **3** by an example that the Lipschitz condition with respect to (p, q) cannot be omitted and, what is more, that there exist continuous f such that (1) has no solution whatsoever (in a vicinity of a given point of the (x, y)-plane).

The existence theorem to be proved for (1) leads to improvements of some of the results of H. Lewy ([14]; cf. the presentation in [6], pp. 487-508). For example, it will be shown (Section **5**) that if $F(x, y, z, p, q, r, s, t)$ is a function of class C^2, and if the partial differential equation

$$F(x, y, z, p, q, r, s, t) = 0, \tag{2}$$

where $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$, is of hyperbolic type, that is, if

$$F_s^2 - 4F_rF_t > 0, \tag{3}$$

then initial data (belonging to a z of class C^3) determine a unique C^3-solution of (2). The conclusion, the existence of a solution z of class C^3, is the same as that of Lewy (cf. [14], p. 189, where it is asserted that z, p, q, r, s, t are of class C^1). But Lewy assumes that F is of class C^3 (instead of being, as above, just of class C^2) and he has the somewhat artificial assumption that the initial data belong to a z of class C^4 (instead of belonging, as above, to a z of only class C^3; cf. [14], p. 187, where it is assumed that the initial data of z, p, q, r, s, t are of class C^2). The conclusion of uniqueness is a corresponding improvement of that of Lewy. (Incidentally, his statement of uniqueness involves solutions of a class which is apparently more

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restricted than the class C^3, but it turns out that all solutions of class C^3 belong to this restricted class.)

When (2) is of the Monge-Ampère type,

$$A + Br + Cs + Dt + E(rt - s^2) = 0, \tag{4}$$

where A, B, C, D, E are functions of the five variables x, y, z, p, q, and (3) is replaced by

$$C^2 - 4BD + 4AE > 0 \text{ and } E \neq 0, \tag{5}$$

it will be shown (Section **6**) that if A, B, C, D, E are of class C^1, then initial data (belonging to a z of class C^2) determine a unique C^2-solution of (4).

Following Lewy's procedure, the theorems involving (2) will be reduced to one on a system (1) by means of an intermediary system of first order of the type

$$\sum_{k=1}^{n} a_{ik} z^k{}_x = b_i, \qquad 1 \leqq i \leqq m (\leqq n), \tag{6}$$

$$\sum_{k=1}^{n} a_{ik} z^k{}_y = b_i, \qquad m+1 \leqq i \leqq n, \tag{7}$$

where the given functions a_{ik}, b_i are functions of $x, y, z^1, \cdots, z^n$ satisfying $\det a_{ik} \neq 0$. For the system (6)-(7) it is sufficient to assume that a_{ik}, b_i are of class C^1, in which case initial data belonging to a $z = (z^1, \cdots, z^n)$ of class C^1 determine a unique C^1-solution z which possesses a continuous second mixed derivative $z_{xy} = z_{yx}$; Section **4**. This is an improvement of a theorem of Lewy ([14], pp. 186-187), where it is assumed that a_{ik}, b_i and the initial data are of class C^2, while the assertion is unchanged. The system (6)-(7) can fail to have solutions if it is only assumed that the given functions a_{ik}, b_i are continuous (instead of being of class C^1), with $\det a_{ik} \neq 0$.

It will also be proved that every C^1-solution z of (6)-(7) has a continuous second mixed derivative $z_{xy} = z_{yx}$. This permits a sharpening of the wording of the above-mentioned uniqueness theorem.

In Part II, the results of Part I just outlined will be applied to two problems of differential geometry; first, the problem of embedding into 3-dimensional Euclidean space a positive-definite line element,

$$ds^2 = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2, \tag{8}$$

of negative curvature; second, the question of transforming (8) into the Tchebychef form

$$ds^2 = dx^2 + 2\cos\phi\, dxdy + dy^2, \qquad \phi = \phi(x, y). \tag{9}$$

Part III adapts the method of Part I to the quasi-linear hyperbolic system for the vector $z = (z^1, \cdots, z^n)$,

$$z_x = fz_y + g,$$

where x and y are scalars, $f = f(x, y, z)$ is a matrix and $g = g(x, y, z)$ a vector. The linear case, where $f = f(x, y)$ is independent of z and g is linear in z, was first treated by Holmgren [11] under the assumption that f and g are of class C^2 (and the system is hyperbolic, that is, f has simple elementary divisors and real characteristic numbers). Perron ([16], pp. 562-564) has shown that Holmgren's method remains applicable if g is not linear in z, but is merely of class C^2 in (x, y, z). Friedrichs ([4], pp. 575-578) has proved that the Holmgren-Perron theory is also valid if $f(x, y)$ and $g(x, y, z)$ are only assumed to be of class C^1. The latter result was stated already by Holmgren ([12], pp. 1-2), whose proof however contains a gap.

Friedrichs has extended the results also to the case in which f is allowed to depend on z but both f and g are subject to the classical C^2-restriction ([4], pp. 582-584; cf. also Cinquini-Cibrario [2] and her papers referred to there). It will be shown in Part III below that it is sufficient that $f(x, y, z)$ be of class C^1 only. In particular, all of the results mentioned above appear as consequences of a single theorem. While the method of Holmgren, Perron and Friedrichs is that of the successive approximations, the proof of the general theorem is based in Part III on a combination of the process of successive approximations and the method of equicontinuous functions, as used in Part I.

Part I.

2. The system (1). A function $z = z(x, y)$ will be said to be of class C^* if $z(x, y)$ is of class C^1 and possesses a continuous second mixed derivative $z_{xy} = z_{yx}$.

(I) *Let* $z = (z^1, \cdots, z^n)$, $p = (p^1, \cdots, p^n)$, $q = (q^1, \cdots, q^n)$ *and* $f = (f^1, \cdots, f^n)$ *be vectors with* n *components. Let* $f = f(x, y, z, p, q)$ *be continuous and bounded on the product space of the rectangle*

$$R\colon\ 0 \leqq x \leqq a, \qquad 0 \leqq y \leqq b \tag{10}$$

and the entire (z, p, q)*-space, and let* f *satisfy a uniform Lipschitz condition with respect to* (p, q) *on this product space. Let* $\sigma(x)$, $\tau(y)$ *be* n*-dimensional vector functions of class* C^1 *on the respective intervals* $0 \leqq x \leqq a$, $0 \leqq y \leqq b$,

and satisfy $\sigma(0) = \tau(0)$. *Then there exists on* (10) *a function* $z = z(x, y)$ *of class* C^* *satisfying* (1) *and*

$$z(x, 0) = \sigma(x) \text{ and } z(0, y) = \tau(y). \tag{11}$$

It will be clear from the proof that if f is not bounded, then it is possible to find an a, where $0 < a \leqq a$, such that the solution $z = z(x, y)$ exists on the square $0 \leqq x \leqq a$, $0 \leqq y \leqq a$. For example, if the absolute value of the components of $\sigma(x)$, $\sigma'(x)$, $\tau(y)$, $\tau'(y)$ are bounded by a number M, and if $C > 0$ is a bound of $|f|$ on the product space of (10) and the $3n$-dimensional parallelopiped $|z^i| \leqq 4M$, $|p^i| \leqq 4M$, $|q^i| \leqq 4M$, where $i = 1, 2, \cdots, n$, then a can be chosen to be min $(a, b, 3M/C, M^{\frac{1}{2}}/C^{\frac{1}{2}})$. In this case, the absolute value of the components of a solution z and its partial derivatives $p = z_x$, $q = z_y$ do not exceed $4M$.

In the simple case when $f(x, y, z, p, q)$ is independent of the derivatives p and q, the assertion (I) reduces to a theorem of Montel [15], p. 282. His proof, as well as that of (I) below, is adapted from the standard proof of the Peano existence theorem in the theory of ordinary differential equations.

Remark. It should be noted that, in the wording of (I), the specification of precisely the class C^* is essential. Of course, every C^1-solution of (1) is a solution of class C^*. On the other hand, the example $z_{xy} = 0$ shows that differential equations of type (1) can have solutions which are not of class C^1. In fact, such a solution is $F(x) + G(y)$ if F, G possess first derivatives having discontinuities (which are, of course, of the second kind by necessity).

Actually (1) need not possess any solution of class C^2 if f satisfies the assumptions of (I). In fact, if $f(x, y)$ is a continuous function in a vicinity of $(x, y) = (0, 0)$, it is clear that the second derivative ϕ_{xx} (or ϕ_{yy}) of the function

$$\phi(x, y) = \int_0^x \int_0^y f(\alpha, \beta)\, d\alpha d\beta$$

will not in general exist; so that $\phi(x, y)$ fails to be of class C^2 (though it is, of course, of class C^*). In addition, $-\phi(x, y)$ will not become of the form $F(x) + G(y)$ if $f(x, y) \not\equiv 0$. Hence, if $f(x, y)$ is just continuous, then for no choice of $F(x)$ and $G(y)$ need the sum $\phi + F + G$ become of class C^2. Since every solution $z(x, y)$ of (1) is such a sum if $f = f(x, y)$, the assertion follows.

Proof of (I). For a fixed m, let $R_{jk} = R_{jk}^m$, where $j, k = 0, \cdots, m - 1$, denote the rectangle

$$(12) \qquad R_{jk} = R_{jk}^{m}\colon\ ja/m \leqq x < (j+1)a/m,\ kb/m \leqq y < (k+1)b/m.$$

Three vector functions $Z = Z_m(x, y)$, $P = P_m(x, y)$, $Q = Q_m(x, y)$ will be defined on R with the properties that Z, P, Q are constant on R_{jk} and that if

$$(13) \qquad z = z_m(x, y) = \sigma(x) + \tau(y) - \sigma(0) + \int_0^x \int_0^y f(u, v, Z(u, v), P(u, v), Q(u, v))\, du dv,$$

then, on R_{jk},

$$(14) \qquad Z(x, y) = z(ja/m, kb/m),$$

$$(15) \qquad P(x, y) = p(ja/m, kb/m), \text{ where } p(x, y) = \lim_{\alpha \to -0} z_x(x + \alpha, y),$$

$$(16) \qquad Q(x, y) = q(ja/m, kb/m), \text{ where } q(x, y) = \lim_{\beta \to -0} z_y(x, y + \beta)$$

(if either $x = 0$ or $y = 0$, the limits defining p and q in (15) and (16) are meant to be replaced by limits as $\alpha \to +0$ and $\beta \to +0$).

In order to obtain the functions Z, P, Q, first put $Z(x, y) = \sigma(0) = \tau(0)$, $P(x, y) = \sigma_x(0)$, $Q(x, y) = \tau_y(0)$ on R_{00}. Then (13) defines $z(x, y)$ as a function of class C^1 on the closure of R_{00}, and the analogues of (14), (15), (16) hold on $R_{jk} = R_{00}$. Define Z, P, Q on R_{10}, R_{01}, R_{11} by (14), (15), (16), respectively, and define z on the closure of these rectangles by (13). Repeat this procedure, defining Z, P, Q on R_{02}, R_{21}, R_{22}, R_{12}, R_{02} by (14), (15, (16) and z on the closures of these rectangles by (13). Continuing in this fashion, the desired functions Z, P, Q, z are defined on R. (In the last step, where either j or k is $m - 1$, the functions Z, P, Q can be defined on the closure of R_{jk}.)

If none of the functions $|\sigma(x)|$, $|\tau(y)|$, $|\sigma'(x)|$, $|\tau'(y)|$, $|f|$ exceeds the number M, then $|z^i|$, $|p^i|$, $|q^i|$ do not exceed $3M + Mab$, $M + Mb$, $M + Ma$, respectively, for $i = 1, \cdots, n$. In particular, bounds for z, p, q can be chosen independently of m. Hence there exists a subsequence of the integers with the property that the corresponding subsequences of the sequences $z_1, z_2, \cdots, p_1, p_2, \cdots$, and $q_1, q_2, \cdots$ converge on a dense set of points of R. Let $z_1, z_2, \cdots$; $p_1, p_2, \cdots$; $q_1, q_2, \cdots$ denote these subsequences. It is clear from (13) and the boundedness of f that $z_1, z_2, \cdots$ is equicontinuous on R. Consequently, $z_1, z_2, \cdots$ converges uniformly to a continuous function $z(x, y)$ on R. The relations (14) show that $Z_1, Z_2, \cdots$ also converges uniformly to $z(x, y)$ on R.

It will be shown that the sequences $p_1, p_2, \cdots$ and $q_1, q_2, \cdots$ also converge uniformly to continuous functions on R. For any (scalar or vector)

function h and for any positive number δ, let $w_h = w_h(\delta)$ denote the least upper bound of $|h(x,y) - h(x^*,y^*)|$ for $|x - x^*| \leqq \delta$, $|y - y^*| \leqq \delta$, where $|h|$ denotes the sum of the absolute values of the components of a vector h. In order to prove the existence of uniform continuous limits for $p_1, p_2, \cdots$ and $q_1, q_2, \cdots$, it is sufficient to show that for every $\epsilon > 0$ there exist a $\delta = \delta_\epsilon > 0$ and an integer $N = N_\epsilon$ such that

$$w_h(\delta) < \epsilon \text{ if } h = p_m, q_m, \text{ and } m \geqq N_\epsilon. \tag{17}$$

To this end, note that if (x,y) is not on any of the lines $x = ja/m$ or $y = jb/m$, where $j = 0, 1, \cdots, m$, then, by (13),

$$p(x,y) = \sigma'(x) + \int_0^y f(x, v, Z(x,v), P(x,v), Q(x,v))dv. \tag{18}$$

Hence if (x^*,y^*) is not on any the lines $x = ja/m$ or $y = jb/m$, and if $|x - x^*| \leqq \delta$, $|y - y^*| \leqq \delta$, then

$$|p(x,y) - p(x^*,y^*)| \leqq w_{\sigma'} + M|y - y^*| + b(w + Kw_P + Kw_Q),$$

where

$$w = \text{l. u. b.} |f(x,y,Z(x,y),P(x,y),Q(x,y)) - f(x^*,y^*,Z(x^*,y^*),P(x,y),Q(x,y))| \tag{19}$$

for $|x - x^*| \leqq \delta$, $|y - y^*| \leqq \delta$, and K is a positive constant implicit in the assumption of a uniform Lipschitz condition on f. Clearly,

$$w_p \leqq w_{\sigma'} + M\delta + b(w + Kw_P + Kw_Q). \tag{20}$$

It is similarly seen that

$$w_q \leqq w_{\tau'} + M\delta + a(w + Kw_P + Kw_Q). \tag{21}$$

The relation (15) shows that $w_P(\delta) \leqq w_P(\delta + 2a/m)$, so that $w_P(\delta) \leqq w_P(2\delta)$ if $2a/m \leqq \delta$. For any function h, $w_h(2\delta) \leqq 2w_h(\delta)$; hence $w_P(\delta) \leqq 2w_p(\delta)$ when $2a/m \leqq \delta$. Similarly, $w_Q(\delta) \leqq 2w_q(\delta)$ if m is sufficiently large. Thus (20) and (21) give, for large m,

$$w_p + w_q \leqq (w_{\sigma'} + w_{\tau'} + 2M\delta + (a+b)w)/(1 - 2(a+b)K), \tag{22}$$

provided that $2(a+b)K < 1$. There is no loss of generality in supposing $2(a+b)K < 1$, for otherwise the rectangle R can be divided into a finite number of sufficiently small rectangles, and theorem (I) applied successively to each of these sub-rectangles (in a suitable order).

In view of the definition (19) of w, the existence, for every $\epsilon > 0$, of a

$\delta = \delta_\epsilon$ and an N_ϵ satisfying (17) follows from (22) and the fact that $Z(x, y) = Z_m(x, y)$ tends, as $m \to \infty$, uniformly on R to a continuous function $z(x, y)$. Consequently, $p_1, p_2, \cdots$ and $q_1, q_2, \cdots$ tend uniformly on R to continuous functions, say $p(x, y)$ and $q(x, y)$. The relations (15) and (16) show that $P_m(x, y) \to p(x, y)$ and $Q_m(x, y) \to q(x, y)$ also hold uniformly on R as $m \to \infty$. It follows from (13) that

$$(23) \qquad z(x, y) = \sigma(x) + \tau(y) - \sigma(0) + \int_0^x \int_0^y f(u, v, z, p, q)\,du\,dv$$

on R, and from (18) that $z_x(x, y) = p(x, y) = \sigma'(x) + \int_0^y f(x, v, z, p, q)\,dv$ if (x, y) is not on any line $x = ja/m$ or $y = kb/m$, where $j, k = 0, 1, \cdots, m$ and $n = 1, 2, \cdots$. By the continuity of the functions involved, $z_x(x, y) = p(x, y)$ on R. Similarly, $z_y(x, y) = q(x, y)$. In view of (23), this completes the proof of (I).

(II) *Let $f(x, y, z, p, q)$ satisfy the conditions of* (I). *Let $\sigma(x)$, $\tau(y)$ be functions of class C^1 for $0 \leqq x \leqq a$, $0 \leqq y \leqq b$, respectively. Let $\Gamma: x = x(u)$, $y = y(u)$, where $0 \leqq u \leqq 1$, be an arc of class C^1 such that $(x(0), y(0)) = (0, 0)$, $(x(1), y(1)) = (a, b)$ and $x'(u) > 0$, $y'(u) > 0$. Then there exists a function $z = z(x, y)$ of class C^* on* (10) *satisfying* (1) *and*

$$(24) \qquad z(x(u), y(u)) = \sigma(x(u)) + \tau(y(u)), \qquad 0 \leqq u \leqq 1.$$

The proof of (II) is similar to that of (I) and will be omitted. If f is not bounded, one obtains a solution $z = z(x, y)$ only in some vicinity of the open arc $\Gamma_0: x = x(u)$, $y = y(u)$, where $0 < u < 1$. For example, if the absolute value of the components of $\sigma(x)$, $\sigma'(x)$, $\tau(y)$, $\tau'(y)$ do not exceed M, and if $|f|$ does not exceed $C > 0$ on the product space of (10) and the $3n$-dimensional parallelopiped $|z^i| \leqq 3M$, $|p^i| \leqq 3M$, $|q^i| \leqq 3M$, where $i = 1, \cdots, n$, then there exists a solution of (1) and (24) on the common part of the two (x, y)-sets

$$(25) \quad \max(0, x(u) - a) \leqq x \leqq \min(a, x(u) + a),\ y = y(u),\ 0 \leqq u \leqq 1,$$

$$(26) \quad x = x(u),\ \max(0, y(u) - a) \leqq y \leqq \min(b, y(u) + a),\ 0 \leqq u \leqq 1,$$

where $a = \min(M^{\frac{1}{2}}/C^{\frac{1}{2}}, 2M/C)$.

3. Counter-examples. In this section, it will be shown that

(i) a solution $z = z(x, y)$ of (1) and (11), whose existence is assured by (I), need not be unique; and that

(ii) (I) becomes false if the Lipschitz condition on f with respect to (p, q) is omitted.

Ad (i). Let $n = 1$ in (I), so that z, p, q, f are scalars. Let $f(x, y, z, p, q) = |z|^{\lambda}$, where $0 < \lambda < 1$, and let $\sigma(x) = 0$, $\tau(y) = 0$. Thus (1) reduces to $z_{xy} = |z|^{\lambda}$ and the initial conditions (11) to $z(x, 0) = 0$, $z(0, y) = 0$. This initial value problem has on the rectangle $R: 0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, more than one solution of class C^*, since it is satisfied by both $z(x, y) \equiv 0$ and $z(x, y) = \mu(xy)^{1/(1-\lambda)}$, where $\mu = (1-\lambda)^{2/(1-\lambda)}$.

Ad (ii). By a solution of (1) is meant a function of class C^* satisfying (1). Consider the case of a scalar equation (1) of the form $z_{xy} = f(x, z_y) + y$. It is clear that this equation has a solution of class C^* if and only if the first order *partial* differential equation

$$z_x = f(x, z) + y \tag{27}$$

has a continuous solution $z(x, y)$, which then possesses a continuous partial derivative with respect to x.

Let it be granted for the moment that there exist continuous functions $f(x, z)$ on $-1 \leqq x \leqq 1$, $-\infty < z < \infty$, with the property that the *ordinary* initial value problem

$$dz/dx = f(x, z), \qquad z(0) = z_0, \tag{28}$$

has more than one solution for *every* z_0, and that if $z^*(x)$, $z_*(x)$ denote the maximum and minimum solutions of (28), then

$$z^*(2^{-j}) \geqq z_0 + c2^{-2j} \text{ and } z_*(2^{-j}) \leqq z_0 - c2^{-2j} \tag{29}$$

hold for $j = 0, 1, \cdots$ and some positive c which is independent of z_0 and j. It will be shown that, for such an f, (27) possesses no continuous solution whatever in any square $|x| \leqq \epsilon$, $|y| \leqq \epsilon$. Consequently, there exist continuous functions f such that (1) has no solution (of class C^*) in any vicinity of a given point.

Suppose (27) has a solution $z = z(x, y)$ on some square $|x| \leqq \epsilon$, $|y| \leqq \epsilon$. In particular, $z(0, y)$ is continuous for $|y| \leqq \epsilon$. For a fixed y, the function $z = z(x, y)$ satisfies the ordinary initial value problem

$$dz/dx = f(x, z) + y, \qquad z(0) = z(0, y).$$

If y is positive, then $z(x, y)$ is not less than the maximal solution of (28), where $z_0 = z(0, y)$. Hence $z(2^{-j}, y) \geqq z(0, y) + c2^{-2j}$ if $2^{-j} \leqq \epsilon$. Similarly, if y is negative, $z(2^{-j}, y) \leqq z(0, y) - c2^{-2j}$. These two inequalities imply that $z(x, y)$ is not continuous at $(x, y) = (2^{-j}, 0)$, since $c > 0$.

Thus, in order to complete the proof of the assertion (ii), it remains to show that there exist functions $f(x, z)$ having the properties specified in connection with (28). It turns out that an $f(x, z)$ defined, in another connection, by van Kampen ([13], p. 151) has the desired properties. His function $f(x, z)$ is defined as follows:

In an (x, z)-plane, consider the curves

$$(30_1) \qquad z = 4n + \alpha(1 - \cos 2\pi x), \text{ where } 0 \leqq x \leqq 1,$$

$$(30_2) \qquad z = 4m + 2 + \beta(1 + \cos 2\pi x), \text{ where } 0 \leqq x \leqq 1$$

depending on the parameters n, m, α, β, where $n, m = 0, \pm 1, \cdots$ and $-1 < \alpha < 1$, $-1 \leqq \beta \leqq 1$. The curves (30_1) belonging to a fixed n and varying α have only the points $(x, z) = (0, 4n)$ and $(1, 4n)$ in common, at which points the derivative z_x is 0. These points are also on the curve (30_2), where $m = n - 1$ and $\beta = 1$. The curves (30_2) belonging to a fixed m and varying β have only the point $(x, z) = (\frac{1}{2}, 4m + 2)$ in common, at which point the derivative z_x is 0. The points just mentioned are the only points common to two curves in the families (30_1)-(30_2); at such a point $z_x = 0$. Through other points (x, z), where $0 \leqq x \leqq 1$, $-\infty < z < \infty$, there passes one and only one curve in the families (30_1)-(30_2).

Let $g(x, z)$, for $0 \leqq x \leqq 1$, $-\infty < z < \infty$, denote the slope z_x of (30_1)-(30_2) at (x, z). Clearly, $g(x, z)$ is a continuous function and satisfies $g(0, z) = g(\frac{1}{2}, z) = g(1, z) = 0$. If z_0 is a number satisfying $4n \leqq z_0 < 4n + 4$, where $n = 0, \pm 1, \cdots$, then the ordinary initial value problem $dz/dx = g(x, z)$, $z(0) = z_0$, has a solution $z = z(x)$ which, for $0 \leqq x \leqq \frac{1}{2}$, is (30_2), where $m = n$ and $4n + 2 + 2\beta = z_0$ (so that $-1 \leqq \beta < 1$), and, for $\frac{1}{2} \leqq x \leqq 1$, is $z(x) = 4n + 2 + (1 + \cos 2\pi x)$. Hence, $z(1) = 4n + 4$.

Define $f(x, z)$ for $-1 \leqq x \leqq 1$, $-\infty < z < \infty$ as follows:

$$f(-x, z) = f(x, z), f(0, z) = 0, \text{ and } f(x, z) = 2^{-j} g(2^j(x - 2^{-j}), 2^{2j} z)$$

if $2^{-j} \leqq x \leqq 2^{-j+1}$, $-\infty < z < \infty$ and $j = 1, 2, \cdots$.

Since $g(0, z) = g(1, z) = 0$, the function $f(x, z)$ is continuous and satisfies $f(2^{-j}, z) = 0$ for $j = 0, 1, 2, \cdots$. If $2^{-k} \leqq x \leqq 2^{-k+1}$, the equation $dz/dx = f(x, z)$ is equivalent to $du/dt = g(t, u)$, where $u = 2^{2k} z$ and $t = 2^k(x - 2^{-k})$. Hence it is seen from the above description of certain solu-

tions of the latter differential equation that through the point $(x, z) = (2^{-k}, z_k)$, where $2^{-2k+2}m \leqq z_k < 2^{-2k+2}(m+1)$ and $m = 0, \pm 1, \cdots$, there is a solution $z(x)$ of $dz/dx = f(x, z)$ satisfying $z(2^{-k+1}) = (m+1)2^{-2k+2}$. Hence this solution can be chosen so as to satisfy $z(2^{-k+2}) = (r+1)2^{-2k+4}$ or $z(2^{-k+2}) = (r+2)2^{-2k+4}$ according as m is of the form $4r$, $4r+1$, $4r+2$ or is of the form $4r+3$. Thus, in any case, $z(2^{-k+2}) \geqq z(2^{-k}) + 2^{-2k+2}$ or, if $k-2$ is denoted by k, simply $z(2^{-k}) \geqq z(2^{-k-2}) + 2^{-2-k}$.

If $z^* = z^*(x)$ is the maximal solution of the initial value problem (28), it is clear that $z^*(2^{-k-2}) \geqq z^*(0)$; hence $z^*(2^{-k}) \geqq z^*(0) + 2^{-2k-2}$. This is the first inequality in (19), with $c = 2^{-2} > 0$. It is proved in the same way that the minimal solution $z_*(x)$ of (28) satisfies the second inequality in (29). This completes the proof of (ii).

4. The system (6)-(7). The existence theorem for the system (6)-(7) will be reduced to that of a system (1).

(III) *Let $a_{ik} = a_{ik}(x, y, z)$, $b_i = b_i(x, y, z)$, where $i, k = 1, \cdots, n$, be functions of class C^1 on the product space of the (x, y)-rectangle* (10) *and the n-dimensional z-space $(-\infty < z^1 < \infty, \cdots, -\infty < z^n < \infty)$, and let*

$$\det a_{ik} \neq 0. \tag{31}$$

Let the arc Γ and the functions $\sigma(x)$, $\tau(y)$ satisfy the conditions of (II). *Then there is a number $a > 0$, such that, on the common part of the sets* (25)-(26), *there exists a $z = z(x, y)$ of class C^* satisfying* (6)-(7) *and* (24).

It will be clear from the proof that one can also obtain the existence of solutions of initial value problems of the type (6)-(7) and (11) on a sufficiently small square $0 \leqq x \leqq a$, $0 \leqq y \leqq a$.

It turns out that the situation in (III) is similar to that in (I) and (II) in that every C^1-solution of (6)-(7) is of class C^*.

(iii) *Let a_{ik}, b_i satisfy the conditions of* (III), *and let the vector $z = z(x, y)$ be a function of class C^1 satisfying* (6)-(7) *on some domain. Then $z(x, y)$ is of class C^*.*

In contrast to (I) or (II), there is a uniqueness theorem corresponding to (III) which does not involve any additional hypothesis.

(III bis) *Under the conditions of* (III), *there cannot exist two distinct C^1-solutions $z = z(x, y)$ of* (6)-(7) *and* (24) *on the common part of* (25) *and* (26), *for any $a > 0$.*

Lewy ([14], pp. 186-187) has proved a uniqueness theorem for the case in which the coefficient functions a_{ik} and b_i, and the initial data Γ, $\sigma(x)$ and $\tau(y)$ are of class C^2, and the class of solutions considered is the class C^*. In view of (iii), Lewy's restriction to the class of C^*-solutions is not more severe than the restriction in (III bis) to the class of C^1-solutions.

Both the existence theorem (III) and the uniqueness theorem (III bis) become false if the assumption that a_{ik}, b_i are of class C^1 is lightened to the assumption that they are continuous. As to (III), this is seen from the example (27) above, which is of type (6)-(7), where $m = n = 1$. (This remark concerning the impossibility of reducing the assumptions on a_{ik}, b_i in (III) is valid if the system (6)-(7) is of the type considered by Lewy ([14], pp. 186-187), where a_{ik}, b_i do not depend on the independent variables x, y. For, under suitable initial conditions, (27) is equivalent to a system of type (6)-(7), namely to $x_u = 1$, $z_u = f(x, z) + y$, $y_v = 1$, where x, y, z are the dependent, and u, v the independent, variables and $n = 3$, $m = 2$.)

As to (III bis), the equation $z_x = |z|^\lambda$, where $0 < \lambda < 1$, which is of type (6)-(7), with $m = n = 1$, has the trivial solution $z(x, y) = 0$, as well as a solution given by $z(x, y) = [(1-\lambda)(x-y)]^{1/(1-\lambda)}$ or $z(x, y) = 0$ according as $x \geqq y$ or $x < y$. Both of these solutions satisfy (24) if Γ is the line $x = y$, and $\sigma(x) = 0$, $\tau(y) = 0$.

Proof of (III). If $z = z(x, y)$ is of class C^* and satisfies (6)-(7), let (6), (7) be differentiated with respect to y, x, respectively. In view of (31), the resulting set of n equations can be solved for the n components of $z_{xy} = z_{yx}$. This leads to a system of equations (1) in which each component of f is of the form $U + V + W$, where U is a quadratic form in the components of z_x and z_y with coefficients which are continuous functions of x, y, z, V is a corresponding linear form, and W is a continuous function of x, y, z. Conversely, if $z = z(x, y)$ is any solution of class C^* of this particular system (1), then $z = z(x, y)$ is also a solution of (6)-(7). Consequently, (III) follows from (II) and the remark made immediately after (II).

This proof shows that (6)-(7) is equivalent to a system (1) if only solutions of class C^* are considered. In view of (iii), the same is true if solutions of class C^1 are considered.

Proof of (iii). Let (x_0, y_0) be a point of (10), and let $\alpha \neq 0$, $\beta \neq 0$ be number such that $z(x, y)$ is defined on the rectangle with the vertices $P_1 = (x_0, y_0)$, $P_2 = (x_0 + \alpha, y_0)$, $P_3 = (x_0 + \alpha, y_0 + \beta)$, $P_4 = (x_0, y_0 + \beta)$. Let (6) be integrated with respect to x from x_0 to $x_0 + \alpha$ (along P_1P_2). Then an integration by parts shows that

$$(32)\quad \sum_{k=1}^{n} a_{ik}(P_1)[z^k]_1{}^2 = -\sum_{k=1}^{n}[a_{ik}]_1{}^2 z^k(P_2) + \int_{P_1P_2}\sum_{k=1}^{n} a_{ikx}z^k dx + \int_{P_1P_2} b_i dx,$$

where $a_{ikx} = \partial a_{ik}/\partial x + \sum_{j=1}^{n}(\partial a_{ik}/\partial z^j)z_x{}^j$ and, for any function h, the bracket is defined by $[h]_1{}^2 = h(P_2) - h(P_1)$. Similarly, an integration of (6) from P_4 to P_3 gives

$$(33)\quad \sum_{k=1}^{n} a_{ik}(P_4)[z^k]_4{}^3 = -\sum_{k=1}^{n}[a_{ik}]_4{}^3 z^k(P_3) + \int_{P_4P_3}\sum_{k=1}^{n} a_{ikx}z^k dx + \int_{P_4P_3} b_i dx,$$

where the left side can be written as

$$\sum_{k=1}^{n} a_{ik}(P_1)[z^k]_4{}^3 + \sum_{k=1}^{n}[a_{ik}]_1{}^4[z^k]_4{}^3.$$

If $z^k(P_2)$ on the right side of (32) is expanded to $z^k(P_3) - [z^k]_2{}^3$, then subtraction of (32) from (33) gives

$$(34)\quad \sum_{k=1}^{n} a_{ik}(P_1)[z^k] = -\sum_{k=1}^{n}[a_{ik}]z^k(P_3) + \Big(\int_{P_4P_3} - \int_{P_1P_2}\Big)a_{ikx}z\,dx$$
$$-\sum_{k=1}^{n}[a_{ik}]_1{}^2[z\,]_2{}^3 - \sum_{k=1}^{n}[a_{ik}]_1{}^4[z^k]_4{}^3 + \Big(\int_{P_4P_3} - \int_{P_1P_2}\Big)b_i dx,$$

where $[h] = h(P_1) - h(P_2) + h(P_3) - h(P_4)$, for any function h, so that $[h] = [h]_4{}^3 - [h]_1{}^2 = [h]_2{}^3 - [h]_1{}^4$.

It will be shown that if the right side of (34) is divided by $\alpha\beta$, then the quotient tends to a limit if first $\alpha \to 0$ and then $\beta \to 0$. This is clear for the last three terms of (34). As to the first two terms on the right of (34), the corresponding quotient has, as $\alpha \to 0$, the limit

$$\beta^{-1}\Big\{\sum_{k=1}^{n}[a_{ikx}]_1{}^4 z^k(P_4) + \sum_{k=1}^{n}[a_{ikx}z^k]_1{}^4\Big\},\ \text{i. e.,}\ \beta^{-1}\sum_{k=1}^{n} a_{ikx}(P_1)[z^k]_1{}^4.$$

But this expression tends to a limit as $\beta \to 0$.

For the purpose of combining equations (34), which resulted from (6), with the corresponding equations which result from (7), it will be verified that the ratio of the right side of (34) to $\alpha\beta$ also tends to a limit if $\beta \to 0$ and then $\alpha \to 0$. To this end, add and subtract the term

$$(35)\qquad -\Big(\int_{P_3P_2} - \int_{P_1P_4}\Big)\sum_{k=1}^{n} a_{iky}z^k dy$$

on the right side of (34). This term, when combined with the first term

on the right of (34), behaves as did the first two terms of (34). The ratio of the last three terms of (34) to $\alpha\beta$ clearly has a limit as $\beta \to 0$ and then $\alpha \to 0$.

Thus there remains to consider the difference of (35) and the second term of (34). This difference is the line integral

$$\int -\sum_{k=1}^{n} z^k (a_{ikx}dx + a_{iky}dy)$$

around the oriented rectangle $P_1P_2P_3P_4$. If the derivatives a_{ikx} and a_{iky} are of class C^1, then Green's formula shows that this line integral is

$$\int\int \sum_{k=1}^{n} (a_{ikx}z_y{}^k - a_{iky}z_x{}^k)\,dxdy,$$

where the double integral is taken over the rectangle $P_1P_2P_3P_4$. But the Lemma in [8], p. 761, shows that this reduction is also valid in the present case. Since the ratio of this double integral to $\alpha\beta$ tends to a limit as $(\alpha, \beta) \to (0, 0)$, this completes the proof of the statement made before (35).

Let (34′) denote the set of equations which results from (7) in the same way as (34) did from (6). Then, by (31), the system of linear equations (34), (34′) can be solved for the components of $[z]$, and so the preceding considerations show that

$$\lim_{\alpha\to 0} \lim_{\beta\to 0} (\alpha\beta)^{-1}[z] = \lim_{\alpha\to 0} \alpha^{-1}(z_y(P_2) - z_y(P_1)) = z_{xy}(P_1)$$

exists and is continuous. This proves (iii).

Proof of (III bis). Suppose that (6)-(7) and (24) have two C^1-solutions, say $z = z(x, y)$ and $z = z(x, y) + \zeta(x, y)$, on the common part of (25) and (26), for some $\alpha > 0$. The difference, ζ, of these solutions vanishes on Γ.

If, as above, $a_{ik} = a_{ik}(x, y, z)$, $b_i = b_i(x, y, z)$, then it follows from (6) that, if $1 \leqq i \leqq m$,

$$\sum_{k=1}^{n} a_{ik}\zeta_x{}^k + \sum_{k=1}^{n} (z_x{}^k + \zeta_x{}^k)(a_{ik}(x, y, z + \zeta) - a_{ik}) = b_i(x, y, z + \zeta) - b_i.$$

Since a_{ik}, b_i are of class C^1, this implies that

$$\sum_{k=1}^{n} a_{ik}\zeta_x{}^k = O(\sum_{k=1}^{n} |\zeta^k|),$$

where the constant implicit in the O-term depends only on the partial derivatives of $z + \zeta$, a_{ik} and b_i. If (x, y) is in the common part of the sets (25)-(26), let the last relation be integrated with respect to x from the point

$(x(u), y(u))$ on Γ, where $y = y(u)$, to the point (x, y). An integration by parts shows that

$$\sum_{k=1}^{n} a_{ik}\zeta^k = \int_{x(u)}^{x} \sum_{k=1}^{n} a_{ikx}\zeta^k dx + \int_{x(u)}^{x} O(\sum_{k=1}^{n} |\zeta^k|)\, dx,$$

since ζ vanishes on Γ. Consequently,

$$\sum_{k=1}^{n} a_{ik}\zeta^k = O(\int_{x(u)}^{x} \sum_{k=1}^{n} |\zeta^k|\, dx) \text{ for } 1 \leqq i \leqq m.$$

Similarly, (7) implies that

$$\sum_{k=1}^{n} a_{ik}\zeta^k = O(\int_{y(v)}^{y} \sum_{k=1}^{n} |\zeta^k|\, dx) \text{ for } m+1 \leqq i \leqq n,$$

where $(x(v), y(v))$ is the point of Γ for which $x(v) = x$. Since (31) shows that the system of n linear equations represented by the last two formula lines can be solved for the n components of ζ, it is seen that

$$|\zeta(x, y)| = nO(\int_{x(u)}^{x} |\zeta|\, dx + \int_{y(v)}^{y} |\zeta|\, dy), \text{ where } |\zeta| = \sum_{k=1}^{n} |\zeta^k|.$$

Hence if μ is the maximum of $|\zeta|$ on the common part of (25) and (26), then $\mu = 2naO(\mu)$.

Since the constant implicit in the last O depends only on the functions z, ζ, a_{ik}, b_i and their partial derivatives, it follows that $\mu = 0$ if a is sufficiently small. In view of the definition of μ, this means that $\zeta(x, y) \equiv 0$, so that the solution of (6)-(7) and (24) is unique on the common part of (25) and (26), if a is sufficiently small. Actually, the last proviso concerning the smallness of a is superfluous, for it is only necessary to divide the common part of (25) and (26) into a finite number of suitable parts and to apply the arguments just used successively on these parts, arranged in a convenient order.

5. The hyperbolic equation (2). The following theorem is the improved version, mentioned in Section **1**, of the corresponding result of Lewy ([14], pp. 187-191).

(IV) *Let* $\Gamma: x = x(u)$, $y = y(u)$, *where* $0 \leqq u \leqq 1$, *be an arc of class* C^3, *and* Γ_0 *the open part of* Γ, $0 < u < 1$. *For* $0 \leqq u \leqq 1$, *let* $z = z(u)$

be of class C^3, and $p = p(u)$, $q = q(u)$ of class C^2, finally $r = r(u)$, $s = s(u)$, $t = t(u)$ of class C^1. In an 8-dimensional open set containing the points $(x(u), y(u), z(u), p(u), q(u), r(u), s(u), t(u))$, where $0 \leqq u \leqq 1$, let $F(x, y, z, p, q, r, s, t)$ be a function of class C^2 satisfying (3). For $0 \leqq u \leqq 1$, let the functions $x = x(u), \cdots, t = t(u)$ satisfy the relations (2),

$$(36)\quad z' = px' + qy', \qquad (37)\quad p' = rx' + sy', \qquad (38)\quad q' = sx' + ty',$$

and

$$(39)\qquad F_r y'^2 - F_s x'y' + F_t x'^2 \neq 0,$$

where $' = d/du$. Then, in a vicinity of Γ_0, there exists one and only one function $z = z(x, y)$ of class C^3 which satisfies (2) and the initial conditions $z(x(u), y(u)) = z(u), \cdots, z_{yy}(x(u), y(u)) = t(u)$, $0 \leqq u \leqq 1$.

Although the statement of (IV) is rather awkward, (IV) is simply the analogue of (II) when z is a scalar. The arc Γ plays the same role in both theorems. The conditions $x'(u) > 0$, $y'(u) > 0$ in (II), which mean that Γ is not in a characteristic direction at any point, are replaced by (39). The initial conditions in (II) are given by (24) in terms of two arbitrary functions $\sigma(x)$, $\tau(y)$ or, equivalently, by the derivatives $\sigma'(x)$, $\tau'(y)$ and quadratures. The same is true in (IV), where, if $p(u)$ and $q(u)$ are given, $z = z(u)$ is determined by a quadrature of (36), and $r = r(u)$, $s = s(u)$, $t = t(u)$ are determined by (2), (37), and (38). In fact, if x, y, z, p, q are considered as functions of u, and r, s, t as unknowns in the equations (2), (37), (38), then (39) expresses the non-vanishing of a Jacobian, thus assuring that r, s, t are uniquely determined as functions of u (if (2), (37), (38) are satisfied for at least one value of u).

The proof of (IV) will not be given, since it becomes identical with that of Lewy's, if use is made of (III), (III bis) above instead of his theorem on systems of the type (6)-(7). (Another reason for not giving the proof of (IV) is that a similar proof will be given in detail for the case of a Monge-Ampère equation in the next section.)

6. The Monge-Ampère equation (4). As is well known (cf., e. g., [5], pp. 42-43), if the hyperbolic differential equation (2) is of the Monge-Ampère type, then the differential equations for (x, y, z, p, q, r, s, t), which define a characteristic strip of (2), can be replaced by differential equations which involve only (x, y, z, p, q). This formal simplification permits a reduction of the degree of differentiability in the assumptions and assertions of (IV) when (2) is of the form (4). This is the content of the following theorem.

(V) *Let* $\Gamma: x = x(u),\ y = y(u)$, *where* $0 \leqq u \leqq 1$, *be an arc of class* C^2, *and* Γ_0 *the open part of* Γ, $0 < u < 1$. *For* $0 \leqq u \leqq 1$, *let* $z = z(u)$ *be a function of class* C^2, *and let* $p = p(u)$, $q = q(u)$ *be functions of class* C^1. *In a 5-dimensional open set containing* $(x(u), \cdots, q(u))$, *where* $0 \leqq u \leqq 1$, *let* A, B, C, D, E *be functions of* x, y, z, p, q, *of class* C^1 *satisfying* (5), *and let* $\lambda_1 = \lambda_1(x, y, z, p, q)$, $\lambda_2 = \lambda_2(x, y, z, p, q)$ *denote the (real, distinct) roots of the quadratic equation*

$$\lambda^2 + C\lambda + BD - AE = 0, \tag{40}$$

so that λ_1, λ_2 *can be chosen to be of class* C^1.

For $0 \leqq u \leqq 1$, *let* $x = x(u), \cdots, q = q(u)$ *satisfy* (36) *and*

$$(\lambda_2 x' + By' + Eq')(Dx' + \lambda_1 y' + Ep') \neq 0. \tag{41}$$

Then, in a vicinity of Γ_0, *there exists one and only one function* $z = z(x, y)$ *of class* C^2 *satisfying* (4) *and the initial conditions* $z(x(u), y(u)) = z(u)$, $z_x(x(u), y(u)) = p(u)$ *and* $z_y(x(u), y(u)) = q(u)$ *for* $0 \leqq u \leqq 1$.

Proof of (V). Consider the system of partial differential equations

$$\lambda_1 x_u + By_u + Eq_u = 0, \tag{42$_1$}$$

$$Dx_u + \lambda_2 y_u + Ep_u = 0, \tag{42$_2$}$$

$$\lambda_2 x_v + By_v + Eq_v = 0, \tag{42$_3$}$$

$$Dx_v + \lambda_1 y_v + Ep_v = 0, \tag{42$_4$}$$

$$px_v + qy_v - z_v = 0 \tag{42$_5$}$$

in the dependent variables $(z^1, z^2, \cdots, z^5) = (x, y, z, p, q)$ and the independent variables u, v (which do not occur explicitly). This system is of the type (6)-(7), where $n = 5$ and $m = 2$. It is readily verified that the determinant of the matrix of the coefficient functions is

$$-E^2(\lambda_1 - \lambda_2)^2 = -E^2(C^2 - 4BD + 4AE)^2,$$

which, by (5), is not 0. The coefficient functions of (42) are of class C^1. Consequently, (III) is applicable if suitable initial conditions are given.

In order to choose such conditions, let Γ^* be the segment $v = u$, $0 \leqq u \leqq 1$, in the (u, v)-plane. Then, corresponding to the initial functions $x = x(u), \cdots, q = q(u)$, there is an a, $0 < a < 1$, such that in the closed hexagonal region H bounded by the six line segments

$$(v = 0,\ 0 \leqq u \leqq a),\quad (v = u - a,\ a \leqq u \leqq 1),\quad (u = a,\ 1 - a \leqq v \leqq 1),$$

$$(v = 1,\ 0 - a \leqq v \leqq 1),\quad (v = u + a,\ 0 \leqq u \leqq 1 - a),\quad (u = 0,\ 0 \leqq u \leqq a),$$

there exist five functions $x = x(u, v), \cdots, q = q(u, v)$ of class C^* which

satisfy (42_1)-(42_5), and which reduce to the given functions $x(u), \cdots, q(u)$ when $v = u$, $0 \leqq u \leqq 1$.

It will be shown that, in the region H, the functions x, y, p, q of (u, v) satisfy the identity

$$p_u x_v + q_u y_v = p_v x_u + q_v y_u. \tag{43}$$

To this end, multiply (42_1), (42_4) by x_v, x_u, respectively, and subtract the resulting two equations. This gives

$$\lambda_2 y_u x_v - \lambda_1 y_v x_u + E(p_u x_v - p_v x_u) = 0.$$

Similarly, (42_1) and (42_3) lead to

$$\lambda_2 y_u x_v - \lambda_1 y_v x_u + E(q_v y_u - q_u y_v) = 0.$$

Since $E \neq 0$ by (5), the last two relations show that

$$p_u x_v - p_v x_u = q_v y_u - q_u y_v, \tag{44}$$

and this is equivalent to (43).

The following analogue of (42_5) will now be verified:

$$p x_u + q y_u - z_u = 0. \tag{45}$$

If $v = u$, then (45) follows from (42_5) and (36). Hence, (45) will be proved on H it is shown that the right side of (45) has, with respect to v, a partial derivative which vanishes identically. Since $x, \cdots, q$ are of class C^*, the right side of (45) has, with respect to v, a partial derivative which is equal to

$$p_v x_u + q_v y_u + p x_{uv} + q y_{uv} - z_{uv}.$$

A differentiation of (42_5) with respect to u and the identity (43) show that the last expression is identically 0 (since $x_{uv} = x_{vu}$, $y_{uv} = y_{vu}$, $z_{uv} = z_{vu}$). This proves (45).

The Jacobian of x, y with respect to u, v does not vanish on Γ^*, that is,

$$x_v y_u - x_u y_v \neq 0 \tag{46}$$

for $v = u$ and $0 \leqq u < 1$. First, if (46) does not hold at some point of Γ^*, then, by (42_1)-(42_4) and $E \neq 0$, the ratios $x_u : x_v$, $y_u : y_v$, $p_u : p_v$, $q_u : q_v$ are identical. But then $x_u : x'$, $y_u : y'$, $p_u : p'$, $q_u : q'$ are identical, where $'$ denotes differentiation in any direction, say along $v = u$. This contradicts (41). Hence (46) holds in the region H if a is sufficiently small.

In view of (46), the linear equations

$$p_u = r x_u + s y_u, \qquad p_v = r x_v + s y_v \tag{47}$$

determine a pair of continuous functions $r = r(u, v)$, $s = s(u, v)$ on H. Similarly,

$$q_u = sx_u + ty_u, \qquad q_v = sx_v + ty_v \tag{48}$$

determine a pair of continuous functions $s = s(u, v)$, $t = (u, v)$ on H. It follows from (44) and (46) that the two definitions (47), (48) of $s = s(u, v)$ are identical.

That the functions $r(u, v), \cdots, t(u, v)$ satisfy (4) follows by standard procedures, which will be indicated here for the sake of completeness. Multiply (42_1), (42_2) by y_u, x_u, respectively, and add. Since (40) shows that $\lambda_1 + \lambda_2 = -C$, the result is

$$By_u^2 - Cx_uy_u + Dx_u^2 + E(p_ux_u + q_uy_u) = 0,$$

or, in view of (47),

$$By_u^2 - Cx_uy_u + Dx_u^2 + E(rx_u^2 + 2sx_uy_u + ty_u^2) = 0. \tag{49}$$

Multiply (42_1) by $Dy_u - Eq_u - \lambda_2 x_u$, and (42_2) by $Bx_u - E_u - \lambda_1 y_u$, and add. Using the relations $\lambda_1 + \lambda_2 = -C$ and $\lambda_1\lambda_2 = BD - AE$, one obtains

$$A(x_u^2 + y_u^2) + (B - D)(p_ux_u - q_uy_u) + C(p_uy_u + q_ux_u) - E(p_u^2 + q_u^2) = 0,$$

after dividing by $E \neq 0$. If p_u, q_u are eliminated from this relation by the use of (47), (48), it follows that

$$(A + Cs)(x_u^2 + y_u^2) + (B - D)(rx_u^2 - ty_u^2) + Cx_uy_u(r + t) - E[s^2(x_u^2 + y_u^2) + r^2x_u^2 + t^2y_u^2 + 2(r + t)sx_uy_u] = 0.$$

If (49) is multiplied by $r + t$ and the product added to the last equation, it is seen that

$$\{A + Br + Cs + Dt + E(rt - s^2)\}(x_u^2 + y_u^2) = 0.$$

Since, by (46), the last factor does not vanish, (4) is satisfied on H.

It follows from (46) that in a 2-dimensional vicinity of the open segment $\Gamma_0^*: v = u$, $0 < u < 1$, the transformation $x = x(u, v)$, $y = y(u, v)$ from (u, v) to (x, y) has an inverse (of class C^1). Hence the existence statement in (V) follows.

The uniqueness statement follows from (III bis) since, as is well known, every C^2-solution of (4) is built up of characteristics; cf. [5], pp. 43-44.

Part II.

7. Embedding a hyperbolic ds^2. Let $(g_{ik}) = (g_{ik}(u, v))$ be a binary, symmetric, positive-definite matrix of class C^2 in a vicinity of the point $(u, v) = (0, 0)$. The line element (8) is called hyperbolic if the curvature $K = K(u, v)$ belonging to (8) is negative. The problem of the existence of a surface $X = X(u, v)$, in 3-dimensional Euclidean space, $X = (x, y, z)$, such that

$$|\, dX \,|^2 = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2, \tag{50}$$

can be reduced to the determination of a function $z = z(u, v)$ such that the line element

$$ds^2 - dz^2 = (g_{11} - z_u^2)du^2 + 2(g_{12} - z_u z_v)dudv + (g_{22} - z_v^2)dv^2 \tag{51}$$

is positive-definite with a curvature 0, and to the determination of solutions of the equation

$$dx^2 + dy^2 = ds^2 - dz^2 \tag{52}$$

for the functions $x = x(u, v)$, $y = y(u, v)$ (Weingarten; cf. [3], pp. 253-254).

When the coefficients g_{ik} are of class C^2, the problem of determination of a $z(u, v)$ such that (51) has zero curvature leads to a Monge-Ampère equation (4), where $p = z_u$, $q = z_v$, $\cdots$, $t = z_{vv}$; cf. e. g. [3], pp. 253-254. The coefficients B, C, D are linear forms in p, q, with coefficients which depend on g_{ik} and their first order partial derivatives. The coefficient E is $-(\det g_{ik})^{\frac{1}{2}} < 0$, and the function A is $Q + K(\det g_{ik})^2$, where K is the curvature of (8) and Q is a quadratic form in p, q with coefficients depending on g_{ik} and their first order partial derivatives; cf. [9], p. 883. Hence, if $p = q = 0$, the right side of the first inequality in (5) is $-4K(\det g_{ik})^{5/2}$, so that (5) holds if (8) is hyperbolic.

Consequently, the existence theorem (V) for a $z = z(u, v)$, with $p(0, 0) = q(0, 0) = 0$, is applicable when the functions g_{ik} are of class C^3 (so that A, B, C, D, E are of class C^1). The theorem (V) shows that in this case there exist functions $z = z(u, v)$ of class C^2 in a vicinity of $(u, v) = (0, 0)$ such that (51) is positive-definite and of curvature 0. A result in [9], p. 882, then implies the existence of functions $x = x(u, v)$, $y = y(u, v)$ of class C^2 in a vicinity of $(u, v) = (0, 0)$ satisfying (52). Hence one obtains the following theorem:

(*) *Let $(g_{ik}(u, v))$ be a binary, symmetric positive-definite matrix of class C^3 in a vicinity of $(u, v) = (0, 0)$. Then there exists a vector function*

$X = (x(u,v), y(u,v), z(u,v))$ of class C^2 in a vicinity of $(u,v) = (0,0)$ with the property that (50) holds.

Remark. The statement (*) remains true if the classes C^3 and C^2 in the assumption and in the assertion are replaced by C^{n+1} and C^n, respectively, where $n > 2$ is arbitrary.

The truth of this remark follows from the corresponding analogues of (II), (III) and (V), analogues which (n in C^n being at least 3) are supplied by the classical theory of characteristics (and which are clear, of course, from the proofs of the above theorems also).

This theorem contrasts with that of [9], pp. 876-877, where (8) is of the elliptic type ($K > 0$), in which case it is sufficient to assume that the g_{ik} have second order partial derivatives satisfying a Hölder condition (instead of assuming that the g_{ik} are of class C^3) in order to assure the existence of an $X(u,v)$ of class C^2 satisfying (50). In the treatment of the hyperbolic case ($K < 0$), we have blundered in [7], [9]; see [10]. The above result and the remark following (*) fill in the gap created by that blunder.

8. Tchebychef nets. The following theorem on the possibility of transforming (8) into the Tchebychef form (9) will be proved:

(†) *In a vicinity of $(u,v) = (0,0)$, let the coefficients* of

$$ds^2 = g_{11}(u,v)\,du^2 + 2g_{12}(u,v)\,du\,dv + g_{22}(u,v)\,dv^2 \tag{53}$$

be functions of class C^1 and have a non-vanishing determinant. Then there exists, in a vicinity of $(x,y) = (0,0)$, a pair of functions

$$u = u(x,y), \qquad v = v(x,y) \tag{54}$$

which are of class C^1, of non-vanishing Jacobian, and such that (53) *becomes identical with*

$$ds^2 = dx^2 + 2\gamma(x,y)\,dx\,dy + dy^2 \tag{55}$$

by virtue of (54).

As observed by Bianchi ([1], vol. I, p. 157), the possibility of transforming (53), when

$$\det g_{ik} \neq 0, \tag{56}$$

into the normal form (55) follows from an existence theorem of classical type if the functions g_{ik} have continuous partial derivatives of first order satisfying a uniform Lipschitz condition. Thus, the point in (†) is that the

imposition of a Lipschitz restriction is claimed to be superfluous. The role of the assumption (56) is that of assuring the existence of the contravariant representation (g^{ik}) of the tensor (g_{ik}), which, combined with the C^1-character of g_{ik}, means that the Christoffel symbols $\Gamma_{jk}{}^i = \Gamma_{jk}{}^i(u, v)$ exist and are continuous. Hence, it is possible to consider the pair of Servant's partial differential equations

$$u^i{}_{xy} = -\Gamma_{\alpha\beta}{}^i(u^1, u^2)\, u^\alpha{}_x u^\beta{}_y, \text{ where } (u^1, u^2) = (u, v), \tag{57}$$

which represent the formal condition for a pair of functions (54) transforming (53) into (55); cf. [1], vol. I, pp. 156-157.

If (56) is omitted, the assertion of (†) becomes false even if the coefficient functions of (53) are analytic. For instance, if (53) is chosen to be the square of $g(v)\,du$, where $g(v)$ is a non-constant, positive, analytic function of v, and if (53) is identical with (55) by virtue of (54), then $gu_x = 1$ and $gu_y = 1$. This implies on the one hand that $u_x = u_y$, which means that u is a function of $x + y$ alone, and on the other hand that $g^{-1}(v)\,(dx + dy)$ is a complete differential, and so $g^{-1}(v)$, therefore v itself, is a function of $x + y$ alone. Hence the Jacobian of (54) must vanish identically, and so (54) cannot be a transformation.

Proof of (†). Since the $\Gamma_{jk}{}^i$ are continuous, (I) is applicable to (57), which is of type (1) with $(z^1, z^2) = (u^1, u^2)$. Let $\epsilon > 0$, and let $\sigma(x) = (\sigma^1(x), \sigma^2(x))$ and $\tau(y) = (\tau^1(y), \tau^2(y))$ be vectors of class C^1 for $|x| \leqq \epsilon$, $|y| \leqq \epsilon$, respectively, satisfying $\sigma(0) = \tau(0) = 0$. Then (I) implies that, if ϵ is sufficiently small, (57) has on the square $|x| \leqq \epsilon$, $|y| \leqq \epsilon$, a solution $(u(x, y), v(x, y)) = (u^1(x, y), u^2(x, y))$ of class C^* satisfying

$$u^i(x, 0) = \sigma^i(x), \qquad u^1(0, y) = \tau^i(y), \qquad (i = 1, 2). \tag{58}$$

Let the initial data $\sigma(x)$, $\tau(y)$ be so chosen that

$$g_{\alpha\beta}(\sigma^1(x), \sigma^2(x))\sigma^{\alpha'}(x)\sigma^{\beta'}(x) \equiv 1, \qquad |x| \leqq \epsilon, \qquad (' = d/dx), \tag{59}$$

$$g_{\alpha\beta}(\tau^1(y), \tau^2(y))\tau^{\alpha'}(y)\tau^{\beta'}(y) \equiv 1, \qquad |y| \leqq \epsilon, \qquad (' = d/dy), \tag{60}$$

and that

$$\sigma^{1'}(0)\tau^{2'}(0) - \sigma^{2'}(0)\tau^{1'}(0) \neq 0. \tag{61}$$

In view of (58) and (61), the Jacobian of (54) does not vanish at $(x, y) = (0, 0)$ and, therefore, for $|x| \leqq \epsilon$, $|y| \leqq \epsilon$ if ϵ is sufficiently small. It remains to show that (53) is transformed into (55), that is, that

$$g_{\alpha\beta}(u^1, u^2)\, u^{\alpha'} u^{\beta'} \equiv 1, \text{ where } u^i = u^i(x, y) \tag{62}$$

and where the primes denote either $\partial/\partial x$ or $\partial/\partial y$.

This will be deduced from the following geometrical interpretation of (57): If $y = y_0$ is fixed, the vectors $(u^1{}_y(x, y_0), u^2{}_y(x, y_0))$ form a parallel sheaf in the sense of Levi-Civita along the arc $u^1 = u^1(x, y_0)$, $u^2 = u^2(x, y_0)$; cf. [1], vol. II, pp. 806-811. Hence, the squared length $g_{\alpha\beta}(u^1, u^2)u^\alpha{}_y u^\beta{}_y$, where $u^i = u^i(x, y_0)$, is independent of x. But if $x = 0$, it is seen from (58) and (60) that this length is 1. Consequently, (62) holds if the primes denote $\partial/\partial y$. Since it is clear that the same is true if the primes denote $\partial/\partial x$, the proof of (†) is complete.

Part III.

9. Quasi-linear systems. The general theorem announced at the end of Section **1** is as follows:

(VI) *Let f and g in the system of partial differential equations*

$$z_x = fz_y + g, \tag{1}$$

where $z = (z^1, \cdots, z^n)$ is a vector, satisfy the following conditions: $f = f(x, y, z)$ is a matrix function of class C^1 on the (x, y, z)-region

$$(2_1) \quad 0 \leqq x \leqq a, \ |y| \leqq b \qquad (2_2) \quad |z^1| \leqq c, \cdots, |z^n| \leqq c,$$

and has simple elementary divisors and real characteristic numbers $\lambda_1(x, y, z), \cdots, \lambda_n(x, y, z)$, while $g = g(x, y, z)$ is a vector function which is continuous and possesses continuous partial derivatives with respect to $y, z^1, \cdots, z^n$ on (2_1)-(2_2). Then there exists a positive $a(\leqq a)$ such that, if the constant K is so chosen that

$$|\lambda_1| \leqq K, \cdots, |\lambda_n| \leqq K \text{ on } (2_1)\text{-}(2_2), \tag{3}$$

the system (1) possesses on the region

$$0 \leqq x \leqq a, \qquad |y| + Kx \leqq b \tag{4}$$

one and only one C^1-solution $z = z(x, y)$ satisfying

$$z(0, y) = 0. \tag{5}$$

It will be clear from the proof that it is possible to give a (lower) estimate for a in terms of bounds of the elements of the matrix f and their partial derivatives and of the components of g. Theorem (VI) implies that (1) and arbitrary initial C^1-data

$$(5') \quad z(0, y) = z(y), \text{ where } |y| \leqq b \text{ and } |z^i(y)| < c, \quad i = 1, \cdots, n,$$

have one and only one C^1-solution on a region of the type (4).

10. Preliminaries. For a vector (or matrix) v, the symbol $|v|$ will denote the absolute value of the largest component (or element) of v.

The assumptions of (VI) for the matrix $f = f(x, y, z)$ imply that there exists on (2_1)-(2_2) a pair of real matrices $H = H(x, y, z)$, $F = F(x, y, z)$ of class C^1 such that H is non-singular, F is diagonal, and

$$Hf = FH, \tag{6}$$

where $|F| \leqq K$ by (3).

The C^1-character of F (hence of H) can be deduced as follows: The diagonal elements of F are the zeros of the invariant factors of $f - \lambda I$, where I is the unit matrix. When f is of class C^1, the invariant factors of $f - \lambda I$ are polynomials in λ with coefficients which are of class C^1 in $(x, y; z)$. The assumption that the elementary divisors of f are simple means that the invariant factors have no multiple zeros, and so their zeros are of class C^1.

The symbol M will be used to denote a constant (not always the same constant) which is a bound for $|v|$ on (2_1)-(2_2), if v is a combination of f, g, F, H, and their partial derivatives.

In the remainder of this section, all functions will be scalar functions.

Let $F(x, y)$ be a function of class C^1 on

$$0 \leqq x \leqq a, \qquad |y| + Kx \leqq b, \tag{7}$$

and suppose that, on (7),

$$|F| \leqq K. \tag{8}$$

Let $\phi(\xi, \eta)$ be the solution of the ordinary initial value problem

$$d\phi/d\xi + F(\xi, \phi) = 0, \qquad \phi(0, \eta) = \eta, \tag{9}$$

and $\psi = \psi(x, y)$ the solution of the implicit equation

$$\phi(x, \psi) = y. \tag{10}$$

The transformation $(x, y) \to (\xi, \eta)$,

$$\xi = x, \qquad \eta = \psi(x, y), \tag{11}$$

is defined and of class C^1 on (7), and has the inverse

$$x = \xi, \qquad y = \phi(\xi, \eta) \tag{12}$$

and the non-vanishing Jacobian

$$\partial(\xi, \eta)/\partial(x, y) = \partial\psi/\partial y = \exp \int_0^x F_y(t, \phi(t, \psi(x, y)))\, dt. \tag{13}$$

If $G(x, y)$ is a continuous function on (7), then the partial intial value problem

$$z_x = Fz_y + G, \qquad z(0, y) = 0 \tag{14}$$

is transformed by (11) into

$$z_\xi = G, \qquad z(0, \eta) = 0. \tag{15}$$

The latter problem has the unique solution

$$z = \int_0^\xi G(t, \phi(t, \eta))\, dt, \tag{16}$$

or, in terms of (x, y),

$$z = \int_0^x G(t, \phi(t, \psi(x, y)))\, dt. \tag{17}$$

If G is of class C^1, then (17), defined on (7), is the unique solution of (14). Under the assumption of mere continuity of G, it cannot be verified and, in general, is not true, that (17) is a solution of (14). If, however, G is of the form

$$G = (H_x - FH_y)U + V, \tag{18}$$

where $H = H(x, y)$, $U = U(x, y)$, $V = V(x, y)$ are of class C^1 on (7), then (17) is of class C^1 on (7) and is the unique solution of (14). (This remark is a simple case of Theorem 4.4 in [4], pp. 570-571). In order to verify this remark, note that (15), (16) can be written as

$$z_\xi = H_\xi U + V, \qquad z(0, \eta) = 0, \tag{19}$$

$$z = \int_0^\xi (H_\xi U + V)\, dt, \tag{20}$$

respectively. Let Δh denote the difference between the value of the function h at the points $(\xi, \eta + \delta)$, (ξ, η). Then (20) shows that

$$\Delta z = \int_0^\xi \{(\Delta H)_\xi U + H_\xi{}^* \Delta U + \Delta V\}\, dt,$$

where the argument of $H_\xi{}^*$ corresponds to $(\xi, \eta + \delta)$. The first term can be integrated by parts and is

$$(\Delta H)U\,|_0^\xi - \int_0^\xi (\Delta H)U_\xi\, dt.$$

This shows that z has a continuous partial derivative with respect to η,

$$z_\eta = H_\eta U\,|_0^\xi + \int_0^\xi \{-H_\eta U_\xi + H_\xi U_\eta + V_\eta\}\,dt. \tag{21}$$

It is seen from (20) and (21) that z is of class C^1 with respect to (ξ, η), hence with respect to (x, y). It is also clear that z is the unique solution of (14), (18).

The derivative z_y can be calculated from (13) and (21),

$$z_y = H_y U + \int_0^x \{H_x U_y - H_y U_x + V_y\} \exp\Big(\int_t^x F_y\,d\tau\Big)\,dt, \tag{22}$$

where the argument of H_x, H_y, U_x, U_y, V_y in the integral is $(t, \phi(t, \psi(x, y)))$, that of F_y is $(\tau, \phi(\tau, \psi(x, y)))$ and, in (22), it is assumed that $U(0, y) = 0$.

11. The semi-linear case. In order to obtain certain inequalities for the proof of (VI), the existence statement in the semi-linear case of (VI), where the matrix f does not depend on the vector z, will be considered first. In this case, the matrices H, F in (6) depend only on (x, y).

Remark. In the application of this section to the proof of (VI), the matrix $f(x, y, z)$ will be made a function of (x, y) alone, by letting z be a function, says $z = \zeta(x, y)$, of class C^1. Then H and F also become functions of (x, y) alone. In accordance with the paragraph following (6), the symbol M will again be used to denote an upper bound of $|H|$ or $|F|$. On the other hand, C (not M) will be used as an upper bound of $|H_x|$, $|H_y|$, $|F_x|$ and $|F_y|$, since these partial derivatives depend not only on the functions f, g but also on the partial derivatives of the function $\zeta(x, y)$. In other words, C can be taken to be $M(1 + B)$, where B is an upper bound of $|\zeta_x|$ and $|\zeta_y|$.

Existence proof for the semi-linear case. This proof is a modification of that of Holmgren [11], pp. 3-8 and Perron [16], pp. 562-564. Since it is assumed that f, H, and F depend only on (x, y), the introduction of the new dependent variable

$$u = Hz \tag{23}$$

changes (1), (5) into

$$u_x = Fu_y + (H_x - FH_y)H^{-1}u + Hg, \qquad u(0, y) = 0, \tag{24}$$

if use is made of the identities (6) and of the relations $HH^{-1}{}_y = -H_y H^{-1}$, $HH^{-1}{}_x = -H_x H^{-1}$.

Consider the successive approximations $u_0(x, y), u_1(x, y), \cdots$ defined as follows: $u_0(x, y) = 0$ and, for $k = 1, 2, \cdots$,

$$u_{kx} = Fu_{ky} + (H_x - FH_y)H^{-1}u_{k-1} + Hg(x, y, H^{-1}u_{k-1}), \tag{25}$$

$$u_k(0, y) = 0. \tag{25 bis}$$

Since F is a diagonal matrix, it is easily verified that each of the n initial value problems for the components of u_k is of the type (14), (18) provided that $u_{k-1}(x, y)$ is a function of class C^1 satisfying $|H^{-1}u_{k-1}| \leqq c$ on (4), where $0 < a = a_{k-1} \leqq a$. Under this proviso, the system (25)-(25 bis) has a unique C^1-solution $u_k = u_k(x, y)$ on (4), with $a = a_{k-1}$, since F satisfies (8). The analogue of (17) shows that u_k satisfies the inequality

$$|u_k| \leqq a \max |(H_x - FH_y)H^{-1}u_{k-1} + Hg(x, y, H^{-1}u_{k-1})|,$$

where max refers to (x, y) on (4). Since $|H^{-1}u_{k-1}| \leqq c$, this can be written as $|u_k| \leqq (MC + M)a$. Hence, if a is chosen so that

$$(MC + M)a \leqq d, \tag{26}$$

where

$$d = c/n \max |H^{-1}| \leqq c/M, \tag{27}$$

then $z_k = H^{-1}u_k$ satisfies $|z_k| \leqq c$. Thus there exists a positive $a(\leqq a)$, independent of k, such that (25)-(25 bis) has on (4) a C^1-solution $u_k(x, y)$ satisfying

$$|u_k| \leqq (MC + M)x \leqq d \qquad (k = 1, 2, \cdots). \tag{28}$$

If $k \geqq 2$, then, according to (25)-(25 bis), the difference $v_k = u_k - u_{k-1}$ is a solution of the initial value problem

$$v_{kx} = Fv_{ky} + G, \qquad v_k(0, y) = 0, \tag{29}$$

where

$$G = (H_x - FH_y)H^{-1}v_{k-1} + H\{g(x, y, H^{-1}u_{k-1}) - g(x, y, H^{-1}u_{k-2})\}.$$

Hence $|G| \leqq (MC + M)|v_{k-1}|$. A comparison of (29) and (14) shows that, in view of the analogue of (17),

$$|v_k| \leqq (MC + M) \int_0^x |v_{k-1}|\, dx.$$

Since $u_0 \equiv 0$ and $v_1 = u_1 - u_0 \equiv u_1$, the inequality (28) implies that $|v_1| \leqq (MC + M)x$. A simple induction then gives $|v_k| \leqq (MC + M)^k x^k/k!$ on (4) for $k = 1, 2, \cdots$. Consequently, $u_0, u_1, \cdots$ converge uniformly on (4) to a continuous function, say $u = u(x, y)$, which satisfies

$$|u| \leqq (MC+M)x. \tag{30}$$

It is clear from (25) that the existence proof for the semi-linear case will be complete if it is verified that the sequences of partial derivatives $u_{0x}, u_{1x}, \cdots$ and $u_{0y}, u_{1y}, \cdots$ converge uniformly on (4). In view of the uniform convergence of $u_0, u_1, \cdots$, it is therefore sufficient to show that the sequences of partial derivatives are uniformly bounded and equicontinuous.

In order to prove their uniform boundedness, note that

$$|u_{kx}| \leqq K|u_{ky}| + MC + M, \tag{31}$$

by (25), (27) and (28). Consequently,

$$\mu_{k1}(x) \leqq K\mu_{k2}(x) + MC + M, \tag{32}$$

if $\mu_{k1}(\beta)$, $\mu_{k2}(\beta)$ denote the maximum of $|u_{kx}|$, $|u_{ky}|$, respectively, on that part of (4) on which $0 \leqq x \leqq \beta (\leqq \alpha)$. An upper estimate of u_{ky} can be obtained from an analogue of (22), where $z = u_k$, $U = H^{-1}u_{k-1}$ and $V = Hg(x, y, H^{-1}u_{k-1})$. The exponential factor in (22) is majorized by $e^{(MC+M)\alpha}$, which does not exceed an M, by (26) and (27). The non-integrated term H_yU of (22) is majorized by $(MC+M)|u_{k-1}| \leqq (MC+M)^2x$ in view of (28). The term H_xU_y under the integral sign has an absolute value which is not greater than $(MC+M)(M\mu_{k-1,2} + MC + M)$, which can be written as $(MC+M)^2 + (MC+M)\mu_{k-1,2}$. Similarly,

$$|H_yU_x| \leqq (MC+M)^2 + (MC+M)\mu_{k-1,1}.$$

Finally, $|V_y| \leqq MC + M + M\mu_{k-1,2}$. Consequently,

$$\mu_{k2}(x) \leqq (MC+M)^2x + (MC+M)\int_0^x (\mu_{k-1,1} + \mu_{k-1,2})\,dt. \tag{33}$$

The relation (32), with k replaced by $k-1$, and (33) imply that

$$\mu_{k2}(x) \leqq (MC+M)^2x + (MC+M)^2\int_0^x \mu_{k-1,2}\,dt.$$

Since $\mu_{0,2}(x) \equiv 0$, a simple induction gives

$$\mu_{k2}(x) \leqq \sum_{j=1}^{k} (MC+M)^{2j}x^j/j! \leqq -1 + \exp(MC+M)^2x. \tag{34}$$

The definitions of μ_{k1} and μ_{k2} and the inequalities (31) and (34) show that the sequences $u_{0x}, u_{1x}, \cdots$ and $u_{0y}, u_{1y}, \cdots$ are uniformly bounded on (4).

There remains the problem of proving the equicontinuity of these sequences.

In view of the uniform convergence of the sequence $u_0, u_1, \cdots$, there exists, for small positive δ, a positive monotone function $w(\delta)$, which tends to 0 as $\delta \to +0$ and exceeds the moduli of continuity of $H, H_x, H_y, F, g, g_x, g_y, u_0, u_1, \cdots$ and of those combinations of the latter functions which will occur below. (By the modulus of continuity of a vector or matrix is meant the maximum of the moduli of continuity of its components or elements.) Hence (25) shows that

$$w_{k1}(\delta, x) \leqq K w_{k2}(\delta, x) + A w(\delta), \tag{35}$$

where A is a constant and $w_{k1}(\delta, \beta)$, $w_{k2}(\delta, \beta)$ are the moduli of continuity of u_{kx}, u_{ky} on that portion of (4) on which $0 \leqq x \leqq \beta (\leqq a)$.

Straightforward estimates, analogous to those leading to (33), give

$$w_{k2}(\delta, x) \leqq A w(\delta) + A \int_0^x (w_{k-1,1} + w_{k-1,2})\, dt \tag{36}$$

with some constant A (which is a bound for certain combinations of H, F, g and their partial derivatives). It follows from the last two formula lines that

$$w_{k2}(\delta, x) \leqq A w(\delta) + A \int_0^x w_{k-1,2}(\delta, t)\, dt$$

(if A is an appropriate constant, independent of k, x and δ). Hence $w_{02} \equiv 0$ implies that

$$w_{k2}(\delta, x) \leqq A w(\delta) \sum_{j=0}^{k} A^j x^j / j! \leqq A w(\delta) e^{Ax}. \tag{37}$$

The definitions of w_{k1}, w_{k2} and the inequalities (35), (37) prove the equicontinuity of $u_{0x}, u_{1x}, \cdots$ and $u_{0y}, u_{1y}, \cdots$ on (4).

This completes the proof of the semi-linear case of (VI).

Remark 1. The number α determining the region (4) on which the solution $u(x, y)$ was proved to exist is subject only to the inequalities $0 < \alpha \leqq a$ and (26).

Remark 2. The inequality (34) shows that the partial derivative u_y of the solution $u = u(x, y)$ of (25)-(25 bis), the existence of which was just proved, satisfies the inequality

$$|u_y(x, y)| \leqq -1 + \exp(MC + M)^2 \tag{38}$$

on (4). Hence, by (25) and (30),

$$|u_x(x, y)| \leqq K\{-1 + \exp(MC + M^2)\alpha\} + (MC + M)^2 \alpha + M. \tag{39}$$

12. Proof of (VI). The existence proof just completed shows that if $z_0(x, y) = 0$, and if $z_{k-1}(x, y)$ is a function of class C^1 satisfying $| z_{k-1} | \leqq c$ on (4), where $a = a_{k-1}$, $0 < a_{k-1} \leqq a$, then the initial value problem

$$z_{kx} = f(x, y, z_{k-1}(x, y))z_{ky} + g(x, y, z_k), \qquad z_k(0, y) = 0 \tag{40}$$

has a C^1-solution $z_k(x, y)$ with the same properties on (4), with some $a = a_k$ and $0 < a_k \leqq a_{k-1}$, for $k = 1, 2, \cdots$. In order to make the inequalities of the last section more readily available, transform (40) by the change of variables (23). Thus (40) becomes

$$u_{kx} = F_k u_{ky} + (H_{kx} - F_k H_{ky})H_k^{-1}u_k + H_k g(x, y, H_k^{-1}u_k), \tag{41}$$

$$u_k(0, y) = 0, \tag{41 bis}$$

where $u_k = H_k z_k$, $F_k = F(x, y, z_{k-1}(x, y))$, and $H_k = H(z, y, z_{k-1}(x, y))$; cf. the transformation from (1) to (25). Remark 1 at the end of the last section shows that $a = a_k$ is subject only to the restrictions $0 < a_k \leqq a_{k-1}$ and (26), where $C = C_{k-1}$ is a bound of the absolute value of the partial derivatives of the elements of the matrices H_k, F_k, and M is a fixed upper bound of certain combinations of f, g, H, F and their partial derivatives (in which x, y, z are considered as independent variables on (2_1)-(2_2)).

In view of the Remark made at the beginning of Section **11**,

$$C_k \leqq M + MB_k, \tag{42}$$

where B_k is any upper bound for $| u_{kx} |$, $| u_{ky} |$ on (4), with $a = a_k$. It will be shown by induction that there exist positive numbers a, B (independent of k) with the properties that

$$a_k \geqq a > 0, \tag{43_k}$$

and

$$| u_{kx} | \leqq B \text{ and } | u_{ky} | \leqq B \tag{44_k}$$

for $k = 0, 1, \cdots$ on (4).

Let the bound M be fixed so large as to satisfy all of the inequalities involving M in Section **11**, and put $B = 2M$. Let $a(\leqq a_0)$ be a positive number satisfying the inequality

$$(MC^* + M)a \leqq d, \text{ where } C^* = M + MB, \qquad d = c/M \tag{45}$$

(cf. (27)), as well as the inequalities

$$-1 + \exp(MC^* + M)^2 a \leqq B, \tag{46}$$

$$K\{-1 \exp(MC^* + M)^2 a\} + (MC^* + M)^2 a + M \leqq B. \tag{47}$$

The inequalities (43_k), (44_k) are trivially satisfied when $k = 0$. Suppose that they hold for some fixed $k \geqq 0$. In view of (42) and the fact that B_k can be taken to be B, it is seen that $C_k \leqq C^*$. It follows from (45) and the sentence following (41 bis) above that a_{k+1} can be chosen so as to satisfy (43_{k+1}). Finally, (44_{k+1}) is a consequence of (38), (39) in Remark 2 at the end of Section **11**, and of (46) and (47). This completes the induction.

Thus the functions $z_0(x, y), z_1(x, y), \cdots$ are defined (and of class C^1) on (4), where $a(> 0)$ is independent of k. In addition, $| z_k | \leqq c$ for $k = 0, 1, \cdots$, and the sequences $z_{0x}, z_{1x}, \cdots$ and $z_{0y}, z_{1y}, \cdots$ are uniformly bounded on (4).

It will be proved that $z_0, z_1, \cdots$ converges uniformly on (4). To this end, let $\zeta_k = z_k - z_{k-1}$. Then ζ_k satisfies

$$\zeta_{kx} = f_k \zeta_{ky} + (\Delta_k f)\zeta_{k-1,y} + \Delta_k g, \qquad \zeta_k(0, y) = 0,$$

where

$$\Delta_k f = f_k - f_{k-1}, \ \Delta_k g = g_k - g_{k-1}, \text{ and } f_k = f(x, y, z_{k-1}(x, y)),$$
$$g_k = g(x, y, z_{k-1}(x, y)).$$

Let $v_k = H_k \zeta_k$, where $H_k = H(x, y, z_{k-1}(x, y))$. Then v_k satisfies the partial differential equation

$$v_{kx} = F_k v_{ky} + (H_{kx} - F_k H_{ky}) H_k^{-1} v_k + H_k\{(\Delta_k f)\zeta_{k-1,y} + \Delta_k g\};$$

cf. the transformation from (1) to (25). Since $| \Delta_k f | \leqq$ Const. $| v_{k-1} |$ and $| \Delta_k g | \leqq$ Const. $| v_{k-1} |$, it follows from an analogue of (17) that

$$| v_k | \leqq a \text{ const. } (| v_k | + | v_{k-1} |),$$

where const. (independent of k) is a bound for combinations of H_k and the partial derivatives of H_k, F_k, g. If $r = a$ const. < 1 then

$$| v_k | \leqq r | v_{k-1} | / (1 - r).$$

Hence, by induction,

$$| v_k | \leqq A[r/(1 - r)]^{k-1},$$

where A is a bound of $| v_1 |$. Consequently, if a is so small that $r/(1 - r) < 1$, then $v_1 + v_2 + \cdots$ is absolutely and uniformly convergent on (4). Hence the same is true of $\zeta_1 + \zeta_2 + \cdots$. In view of the definition of ζ_k, this proves that the sequence $z_0, z_1, \cdots$ has a uniform limit, say $z(x, y)$, on (4).

It is clear that the proof of the existence statement in (VI) will be complete if it is shown that the sequences $z_{0x}, z_{1x}, \cdots$ and $z_{0y}, z_{1y}, \cdots$ are uniformly convergent on (4). Since these sequences are uniformly bounded and $z_0, z_1, \cdots$ is uniformly convergent, it is sufficient to prove that $z_{0x}, z_{1x}, \cdots$ and $z_{0y}, z_{1y}, \cdots$ are equicontinuous or, what is the same thing, that $u_{0x}, u_{1x}, \cdots$

and $u_{0y}, u_{1y}, \cdots$ are equicontinuous. But the proof of this fact is analogous to the corresponding proof in Section 11 (if $a > 0$ is sufficiently small) and will therefore be omitted.

The proof of the uniqueness statement in (*) follows the standard arguments which are applicable when successive approximations converge.

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ON PICARD VARIETIES.*

By André Weil.

In an important recent paper [7], J. Igusa has defined two abelian varieties attached to a non-singular algebraic subvariety V of a complex projective space, and proved some basic results concerning these varieties and their relationship to each other. In the language of the Italian school, these varieties are those attached respectively to the periods of the simple integrals of the first kind on V, and to the continuous systems on V. It is my purpose here to present a somewhat different approach to the same problem, leading to a proof of Igusa's theorems under more general conditions, and to further results in the same direction. I shall avail myself freely of some of Igusa's ideas, but shall not otherwise make any use of his paper.

§ I. Igusa's duality theorems.

1. A vector space A of dimension q over the field $\boldsymbol{C}$ of complex numbers is at the same time a vector-space, which we shall denote by A_0, of dimension $2q$ over the field $\boldsymbol{R}$ of real numbers; the scalar multiplication by i in A is an automorphism J of A_0, satisfying $J^2 = -I$, where I is the identical automorphism of A_0. Conversely, if A_0 is a vector-space of dimension $2q$ over $\boldsymbol{R}$, and J is a linear mapping of A_0 into itself, satisfying $J^2 = -I$, we can define on A_0 a vector-space structure over $\boldsymbol{C}$ by putting $(\alpha + i\beta)x = \alpha x + \beta Jx$ for any $\alpha \in \boldsymbol{R}$, $\beta \in \boldsymbol{R}$; with this additional structure, A_0 will be denoted by A. Linear forms on A_0 will be called real-linear; linear forms on A will be called complex-linear. The complex-linear forms on A are those complex-valued real-linear forms $L(x)$ on A_0 which satisfy $L(Jx) = iL(x)$; if $M(x)$ is any real-valued real-linear form on A_0, then there is one and only one complex-linear form on A with the real part $M(x)$, viz. $L(x) = M(x) - iM(Jx)$.

A and A_0 being as above, let A'_0 be the dual space of A_0; denote by $\langle x', x\rangle$ the canonical bilinear form on $A'_0 \times A_0$. Put $J' = {}^tJ^{-1} = -{}^tJ$; as we have $J'^2 = -I'$, where I' is the identical automorphism of A'_0, J' can be used to make of A'_0 a complex vector-space A'. Put

$$B(x', x) = \tfrac{1}{2}[\langle x', x\rangle - i\langle x', Jx\rangle] = \tfrac{1}{2}[\langle x', x\rangle + i\langle J'x', x\rangle];$$

* Received December 21, 1951.

for each $a' \varepsilon A'$, $B(a', x)$ is a complex-linear form on A, and every such form can be so written in one and only one way; similarly, for each $a \varepsilon A$, $\overline{B(x', a)}$ is a complex-linear form on A', and every such form can be so written in one and only one way. We shall say that the sesquilinear form $B(x', x)$ defines A' as the antidual space of A.

2. By a *complex torus* Θ, of complex dimension q, we shall understand a torus of real dimension $2q$, with a translation-invariant complex-analytic structure. The universal covering group of Θ is then a vector-space of dimension $2q$ over $\boldsymbol{R}$ with a translation-invariant complex-analytic structure, or, what is the same thing, a vector-space A of dimension q over $\boldsymbol{C}$. In other words, a complex torus Θ is the quotient of a vector-space A of dimension q over $\boldsymbol{C}$ by a discrete subgroup Δ of rank $2q$ of A. If $\Theta_1 = A_1/\Delta_1$, $\Theta_2 = A_2/\Delta_2$ are two such complex toruses, a complex-analytic homomorphism of Θ_1 into Θ_2 will be given by a complex-linear mapping L of A_1 into A_2 such that $L(\Delta_1) \subset \Delta_2$. If Θ_1, Θ_2 have the same dimension, and if there exists an *invertible* complex-linear mapping L of A_1 into A_2 such that $L(\Delta_1) \subset \Delta_2$, then we say that Θ_1, Θ_2 are *isogenous*; as L must then map Δ_1 isomorphically onto a subgroup of finite index of Δ_2, there will then be an integer $n \neq 0$ such that nL^{-1} maps Δ_2 into Δ_1; it follows that isogeneity is an equivalence relation between complex toruses (to this relation, an unfortunate tradition had attached the word "isomorphism," which of course is here restored to its proper meaning).

3. Let Θ be a complex torus, of complex dimension q; let A be the universal covering group of Θ, considered as above as a vector-space of dimension q over $\boldsymbol{C}$. Let A_0 be the underlying real vector-space of A, of dimension $2q$ over $\boldsymbol{R}$, and let J be the automorphism of A_0 defined by the scalar multiplication by i on A. Let Δ be the kernel of the canonical homomorphism of A onto Θ, so that $\Theta = A/\Delta$. Let A' be the antidual space to A, as defined in no. 1, so that its underlying real vector-space A'_0 is the dual of A_0, and that A' is determined over A'_0 by the automorphism $J' = {}^tJ^{-1} = -{}^tJ$. Let Δ' be the subgroup of A'_0 associated with the subgroup Δ of A_0, i. e. consisting of all the elements r' of A'_0 such that $\langle r', r \rangle \equiv 0 \bmod 1$ for all $r \varepsilon \Delta$. Then Δ' is a discrete subgroup of rank $2q$ of A'; and $\Theta' = A'/\Delta'$ is a complex torus, which will be called *dual* to Θ. It is clear that Θ is then dual to Θ'

Let $\Theta_1 = A_1/\Delta_1$, $\Theta_2 = A_2/\Delta_2$ be two complex toruses, and Θ'_1, Θ'_2 their duals. Consider a complex-analytic homomorphism of Θ_1 into Θ_2, given as above by a complex-linear mapping L of A_1 into A_2, such that $L(\Delta_1) \subset \Delta_2$.

But, if A_1, A_2 are defined by the underlying real vector-spaces and the automorphisms J_1, J_2 of these corresponding to the scalar multiplication by i, a complex-linear mapping of A_1 into A_2 is the same as a real-linear mapping L_0 satisfying $L_0J_1 = J_2L_0$; as this implies ${}^tL_0(-{}^tJ_2) = (-{}^tJ_1){}^tL_0$, it follows that ${}^tL_0$ is a complex-linear mapping of A'_2 into A'_1; and one verifies at once that ${}^tL_0(\Delta'_2) \subset \Delta'_1$. Therefore ${}^tL_0$ defines a complex-analytic homomorphism of Θ'_2 into Θ'_1.

4. With the same notations as above, let $\Theta = A/\Delta$ be a complex torus of complex dimension q. Let Γ be the group of all representations of the additive group Δ into the multiplicative group $\boldsymbol{C}^*$ of complex numbers $\neq 0$; let Γ_0 be the subgroup of the elements μ of Γ such that $|\mu(r)| = 1$ for all $r \in \Delta$, i. e. the character-group of Δ. As Δ is a free abelian group of rank $2q$, Γ_0 is a torus of real dimension $2q$, and Γ is isomorphic to $(\boldsymbol{C}^*)^{2q}$. Elements of Γ will be called *multiplicator-sets*. By a multiplicative function belonging to Θ, with the multiplicator-set μ, we shall understand a complex-valued function ϕ on A, not everywhere 0 or ∞, such that $\phi(x + r) = \phi(x)\mu(r)$ for all $x \in A$, $r \in \Delta$. A multiplicative function which is an entire (i. e., everywhere holomorphic) function of the complex coordinates on A, and is nowhere 0, must be of the form [1] $e[L(x) + c]$, where c is a constant and $L(x)$ a complex-linear form on A; the multiplicator-set $e[L(r)]$ of any such function will be called *trivial*; we shall denote by γ the group of all trivial multiplicator-sets.

If $\mu \in \Gamma$, $\log|\mu(r)|$ is a representation of the additive group Δ into the additive group $\boldsymbol{R}$; this can be extended in one and only one way to a real-valued real-linear form $M(x)$ on A_0; let $L(x) = M(x) - iM(Jx)$ be the complex-linear form on A with the real part $M(x)$. Then $\nu(r) = e^{L(r)}$ is a trivial multiplicator-set, satisfying $|\nu(r)| = |\mu(r)|$ for all $r \in \Delta$, and it is the only such set. Hence every coset of γ in Γ has one and only one element in Γ_0, i. e. we have $\Gamma = \gamma \times \Gamma_0$; and Γ/γ may be canonically identified with Γ_0.

Let A', A'_0, Δ' be as in no. 3. Each element of Γ_0 can be written as $e(\langle a', r\rangle)$, with $a' \in A'$; this defines therefore a homomorphism of A' onto Γ_0, the kernel of which is Δ', hence an isomorphism between Γ_0 and A'/Δ'. Thus we have canonical isomorphisms between Γ/γ, Γ_0, and the underlying real torus of $\Theta' = A'/\Delta'$; we can use these to transport to Γ/γ and Γ_0 the complex torus structure of Θ', and to identify these three toruses with one another.

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[1] Here and in the rest of this paper we put $e(t) = e^{2\pi it}$.

5. On Γ, one can define a complex-analytic structure in an obvious manner, viz. by the condition that, for each $r \in \Delta$, $\mu(r)$, as a function of $\mu \in \Gamma$, should be a holomorphic function on Γ. We now prove that the canonical homomorphism of Γ onto Γ/γ is complex-analytic, when the complex structure on Γ/γ is defined as in no. 4, and that this complex structure on Γ/γ is the only one for which this is so. In fact, let B' be the vector-space of dimension $2q$ over $\boldsymbol{C}$, derived from A'_0 by extending the field of scalars of A'_0 from $\boldsymbol{R}$ to $\boldsymbol{C}$; every element of B' can be written as $z' = x' + iy'$, with $x' \in A'_0$, $y' \in A'_0$; if we write $\mu_{z'}(r) = e(\langle x', r\rangle + i\langle y', r\rangle)$, the mapping $z' \to \mu_{z'}$ is a complex-analytic homomorphism of B' onto Γ. Now the element ν of γ which satisfies $|\nu(r)| = |\mu_{z'}(r)|$ for all $r \in \Delta$ is $\nu(r) = e(\langle y', Jr\rangle + i\langle y', r\rangle)$; hence the element of Γ_0 which belongs to the same coset of γ as $\mu_{z'}$ is $e(\langle x', r\rangle - \langle y', Jr\rangle) = e(\langle x' + J'y', r\rangle)$. Now denote that coset by $f(z')$; f is a homomorphism of B' onto Γ/γ. Call ω the canonical homomorphism of A' onto $\Theta' = A'/\Delta'$; and, for $z' = x' + iy'$, put $g(z') = x' + J'y'$; Θ' and Γ/γ being identified as in no. 4, what we have just shown amounts to $f = \omega \circ g$. As we have $g(iz') = -y' + J'x' = J'(x' + J'y')$, g is a complex-analytic homomorphism; so is ω; the same, therefore, is true of f. Conversely, if J'_1 is an automorphism of A'_0, defining on A'_0 a complex structure such that f, hence also g, are complex-analytic mappings, then we must have $g(iz') = J'_1 g(z')$, hence $J'_1 = J'$.

6. From the theory of abelian varieties and of theta-functions ([8], [9]; cf. [11]), we borrow the following facts. In order that there should be q algebraically independent meromorphic functions on the complex torus $\Theta = A/\Delta$, it is necessary and sufficient that there should exist a real-valued alternating bilinear form $E(x, y)$ on $A_0 \times A_0$, such that $E(r, s) \equiv 0 \bmod 1$ whenever $r \in \Delta$, $s \in \Delta$, and that the bilinear form $F(x, y) = E(Jx, y)$ is symmetric and positive-definite (i. e. satisfies $F(x, x) > 0$ for all $x \neq 0$). Such a form E will be called a *Riemann form* for Θ; if we write it as $E(x, y) = \langle Ex, y\rangle$, where E is a linear mapping of A_0 into A'_0, the condition that it should be alternating is expressed by ${}^tE = -E$; this being assumed, we have $F(x, y) = \langle EJx, y\rangle = -\langle Ey, Jx\rangle$, and the condition that F should be symmetric is expressed by $EJ = {}^t(EJ) = J'E$, hence it is equivalent to the condition that E should be a complex-linear mapping of A into A'. The condition $E(r, s) \equiv 0 \bmod 1$ for $r \in \Delta$, $s \in \Delta$, is equivalent to $E(\Delta) \subset \Delta'$. Hence E defines a complex-analytic homomorphism of Θ into its dual Θ'. Finally, if F is positive-definite, we must have $Ex \neq 0$ for $x \neq 0$, hence E is invertible. Therefore the existence of a Riemann form implies that Θ and Θ' are isogenous; if moreover E maps Δ onto Δ', they will be isomorphic, but they need not be so otherwise.

A Riemann form E being given, one can define (by the explicit con-

struction of so-called theta-series) theta-functions belonging to Θ and E, i. e. entire functions ϑ of the complex coordinates on A, satisfying periodicity conditions which, suitably normalized, may be assumed to be

$$\vartheta(x+r) = \vartheta(x)e[-\tfrac{1}{2}E(r,x) + \tfrac{i}{2}E(r,Jx) - \tfrac{1}{4}E^0(r,r) + \tfrac{i}{4}E(r,Jr)]$$

for all $x \in A$, $r \in \Delta$, where $E^0(x,y)$ is a symmetric bilinear form on $A_0 \times A_0$ such that $E^0(r,s) \equiv E(r,s) \bmod 2$ for all $r \in \Delta$, $s \in \Delta$. Then, for any $a \in A$, the function

$$\vartheta(x+a)\vartheta(x)^{-1}e[\tfrac{1}{2}E(a,x) - \tfrac{i}{2}E(a,Jx)]$$

is a multiplicative meromorphic function of the complex coordinates on A, with the multiplicator-set $e[E(a,r)] = e(\langle Ea, r\rangle)$. As E maps A_0 onto A'_0, this, in conjunction with the results of no. 4, shows that there exist meromorphic multiplicative functions with arbitrarily given multiplicator-sets.

Now consider a maximal set of linearly independent theta-functions $\vartheta_j(x)$ belonging to Θ and the Riemann form NE, N being an integer > 0. These must be in finite number; and, if they are taken as homogeneous coordinates of a point $P(x)$ in a suitable projective space, we have $P(x+r) = P(x)$ for $r \in \Delta$, so that the mapping $x \to P(x)$ induces a complex-analytic mapping Φ of Θ into that projective space. By a theorem of Lefschetz [8], as soon as N is large enough, Φ is a one-to-one mapping of Θ onto an algebraic variety $\Phi(\Theta)$, and its Jacobian matrix has everywhere the rank q, so that $\Phi(\Theta)$ is without multiple points; Φ can then be used to identify Θ with $\Phi(\Theta)$. For that reason, we shall say that a complex torus Θ is an *abelian variety* whenever there exists a Riemann form for Θ.

If Θ is an abelian variety, so is every complex torus isogenous to Θ, hence in particular the dual torus Θ'. In fact, let $\Theta_1 = A_1/\Delta_1$ be isogenous to Θ; then there is an invertible complex-linear mapping L of A_1 onto A, such that $L(\Delta_1) \subset \Delta$. Let $E(x,y)$ be a Riemann form for Θ; then the bilinear form $E(Lx_1, Ly_1)$ is alternating, and is $\equiv 0 \bmod 1$ for $x_1 \in \Delta_1$, $y_1 \in \Delta_1$; also, as $LJ_1 = JL$, we have

$$E(LJ_1x_1, Ly_1) = E(JLx_1, Ly_1) = F(Lx_1, Ly_1),$$

where F is defined as before; this is symmetric, and > 0 whenever $Lx_1 = Ly_1 \neq 0$, hence, since L is invertible, whenever $x_1 = y_1 \neq 0$.

7. Now let V be a Kähler manifold, i. e. a connected compact complex-analytic manifold with a Kähler metric; I shall use the notations of a previous note of mine ([10][2]; cf. [3], Part II).

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[2] In that note, one should read $\Omega = \frac{i}{2}\Sigma_\nu \omega_\nu \wedge \bar{\omega}_\nu$ instead of $\Omega = \Sigma_\nu \omega_\nu \wedge \bar{\omega}_\nu$ on p. 110, line 7 from bottom.

If G is the fundamental group of V, and G' the commutator-group of G, $H = G/G'$ is the one-dimensional homology group of V with integral coefficients; it is the direct product of its torsion group T, i. e. of the finite group of its elements of finite order, and of a free abelian group, whose rank, by Hodge's theorems, is an even number $2q$. If we consider H as a module over the ring of integers, and extend [3] its ring of operators to $\boldsymbol{R}$, we obtain a vector-space A_0 of dimension $2q$ over $\boldsymbol{R}$, which is the one-dimensional homology group of V with real coefficients; we shall denote by λ the canonical homomorphism of H into A_0; the kernel of λ is T, and the image $\Delta = \lambda(H)$ of H by λ is a discrete subgroup of rank $2q$ of A_0.

Let $\hat{V}$ be the covering manifold of V belonging to the subgroup G' of G; every element σ of H determines an automorphism of $\hat{V}$, transforming each point $\hat{M}$ of $\hat{V}$ into a point $\sigma\hat{M}$ lying over the same point M of V as $\hat{M}$. If θ is any closed differential form of degree 1 on V, and $\hat{P}$, $\hat{M}$ are two points on $\hat{V}$, the integral $\int_{\hat{P}}^{\hat{M}} \theta$ (taken along any differentiable path from $\hat{P}$ to $\hat{M}$ on $\hat{V}$) defines, when $\hat{P}$ is fixed, a function $f(\hat{M})$ such that $df = \theta$; we have simplified notations here by writing θ also for the inverse image of θ on $\hat{V}$. For any $\sigma \in H$, $p_\sigma = f(\sigma\hat{M}) - f(\hat{M})$ is a constant, which we can also write as $p_\sigma = \int_\sigma \theta$, the integral being taken along any closed differentiable path of class σ on V. Then $\sigma \to p_\sigma$ is a representation of H into the additive group $\boldsymbol{R}$; this must be 0 on T, and it can be extended, in one and only one way, to a linear form on A_0; this form will be denoted by $p(\theta, x)$, so that we have $p_\sigma = \int_\sigma \theta = p[\theta, \lambda(\sigma)]$.

Now let B be the vector-space of real harmonic differential forms of degree 1 on V; then $p(\theta, x)$, as a function of $(\theta, x) \in B \times A_0$, is a bilinear form, which, by Hodge's fundamental theorem, determines a duality between B and A_0; hence we may use it to identify B with the dual space A'_0 of A_0, i. e. with the one-dimensional cohomology group of V with real coefficients; and we write, from now on, $\langle \theta, x \rangle$ instead of $p(\theta, x)$ when θ is harmonic, i. e. when $\theta \in A'_0$.

If θ is harmonic, and if $\hat{P}$, $\hat{M}$ are two points on $\hat{V}$, the integral $\int_{\hat{P}}^{\hat{M}} \theta$ is a real number depending linearly upon θ, which, when $\hat{P}$ is fixed, can be written as $\langle \theta, \hat{F}(\hat{M}) \rangle$, with $\hat{F}(\hat{M}) \in A_0$. Then $\hat{F}$ is a mapping of $\hat{V}$ into A_0, satisfying $\langle \theta, \hat{F}(\sigma\hat{M}) - \hat{F}(\hat{M}) \rangle = \int_\sigma \theta = \langle \theta, \lambda(\sigma) \rangle$, i. e. $\hat{F}(\sigma\hat{M}) = \hat{F}(\hat{M}) + \lambda(\sigma)$,

[3] Cf. N. Bourbaki, *Algèbre*, Chap. III, § 2, no. 1.

hence $\hat{F}(\sigma\hat{M}) \equiv \hat{F}(\hat{M}) \bmod \Delta$. Therefore $\hat{F}$ determines a mapping F of V into the torus A_0/Δ; $\hat{F}$ and F, or any mappings derived from these by the addition of constants, will be called the *canonical mappings* of $\hat{V}$ into A_0, and of V into A_0/Δ, respectively.

So far we have made no use of the complex structure of V. Now we introduce the operator C derived from this structure (cf. [10]); this transforms real differential forms into real differential forms of the same degree, harmonic forms into harmonic forms; hence it induces an automorphism on the space A'_0 of real harmonic forms of degree 1, whose transpose will be denoted by J. As we have $C(C\theta) = -\theta$ for any form of degree 1, we have $J^2 = -I$. Furthermore, if f is any holomorphic function in the neighborhood of a point of V, then we have $C(df) = i \cdot df$, $C(\overline{df}) = -i \cdot \overline{df}$; conversely, any closed form ζ of degree 1, satisfying $C\zeta = i\zeta$, is holomorphic, i. e. it is locally the differential of a holomorphic function. In particular, if θ is a real harmonic form, the form $\zeta = \theta - iC\theta$ is closed and satisfies $C\zeta = i\zeta$, hence it is everywhere holomorphic, i. e. it is a simple differential of the first kind; since, conversely, the real part of such a form is harmonic, this implies that the space A'_0 of real harmonic forms of degree 1 depends only upon the complex structure of V, and not upon the choice of a Kähler metric (provided one exists).

Now we use the automorphism J of A_0 in order to convert A_0 into a vector-space A of dimension q over $\boldsymbol{C}$, so that $\Theta = A/\Delta$ is then a complex torus. As any complex-linear form in A can be written as

$$L(x) = \langle \theta, x \rangle - i\langle \theta, Jx \rangle = \langle \theta, x \rangle - i\langle C\theta, x \rangle,$$

with some $\theta \in A'_0$, we have, for such a form, $L[\hat{F}(\hat{M})] = \int_{\hat{P}}^{\hat{M}} \zeta$ with $\zeta = \theta - iC\theta$, which shows that $L[\hat{F}(\hat{M})]$ is then a holomorphic function on $\hat{V}$. Therefore *$\hat{F}$ and F are complex-analytic mappings* of $\hat{V}$ into A, and of V into Θ, respectively.

8. Let Ω be the differential form of degree 2 on V canonically associated with the given Kähler metric (cf. [10]). If all the periods of Ω are integers (i. e., if Ω is homologous to an integral cocycle), we shall say that V satisfies the Hodge condition, or that it is a Hodge manifold. A recent theorem of Hodge [6] states that, when that is so, the complex torus Θ associated with V is an abelian variety. In our notation, Hodge's proof for this is as follows. Let $*$ be the usual operator, transforming forms of degree d into forms of degree $2n - d$ if n is the complex dimension of V (cf. [10]).

If θ is any differential form of degree 1, one verifies easily that $(C\theta)\Omega^{n-1} = -(n-1)! * \theta$, multiplication being of course the exterior multiplication of differential forms. Put now:

$$E'(\theta, \eta) = -\int_V \theta\eta\Omega^{n-1};$$

this is an alternating bilinear form on $A'_0 \times A'_0$. Call Δ' the subgroup of A'_0 associated with the subgroup Δ of A_0; it consists of those harmonic forms, all of whose periods are integers. As the periods of Ω are integers, one of de Rham's theorems shows that $E'(\theta, \eta) \equiv 0 \bmod 1$ for $\theta \in \Delta', \eta \in \Delta'$. If J' is defined as usual by $J' = -{}^tJ$, we have $J'\theta = -C\theta$ for $\theta \in A'_0$, hence

$$E'(J'\theta, \eta) = (n-1)! \int_V \eta \cdot (*\theta).$$

As this is symmetric and positive-definite, we see that E' is a Riemann form for Θ', so that Θ', hence also Θ, are abelian varieties. We shall call Θ' *the Picard variety*, and Θ *the dual Picard variety, of* V.

9. Henceforward, let V be a Hodge manifold. Let all notations have the same meaning as above; in particular, we still denote by λ the canonical homomorphism of H into A_0, and by $\hat{F}$, F the canonical mappings of $\hat{V}$ into A, and of V into Θ. Let ϵ be any representation of H into the multiplicative group $\boldsymbol{C}^*$; by a multiplicative function on V, with the multiplicator-set ϵ, we shall understand a complex-valued function ϕ on V, not everywhere 0 or ∞, such that $\phi(\sigma\hat{M}) = \phi(\hat{M})\epsilon(\sigma)$ for all $\hat{M} \in \hat{V}$, $\sigma \in H$. In particular, if ψ is a multiplicative function belonging to Θ, with the multiplicator-set μ, $\psi[\hat{F}(\hat{M})]$, provided it is not everywhere 0 or ∞, is a multiplicative function on $\hat{V}$ with the multiplicator-set $\epsilon = \mu \circ \lambda$; any multiplicator-set ϵ of that form will be called *special*, and any multiplicative function on $\hat{V}$ with such a multiplicator-set will be called *special*. A multiplicator-set ϵ is special if and only if it is equal to 1 on the torsion-group T. The mapping $\mu \to \mu \circ \lambda$ is an isomorphism of the group Γ of multiplicator-sets belonging to Θ onto the group Γ' of special multiplicator-sets for V; we shall denote by Γ'_0, γ' the images, by this isomorphism, of the subgroups Γ_0, γ of Γ as defined in no. 4; elements of γ' will be called the *trivial* multiplicator-sets for V.

If ψ is any holomorphic multiplicative function without zeros, belonging to Θ, and μ is its multiplicator-set, then $\mu \in \gamma, \mu \circ \lambda \in \gamma'$, and $\psi[\hat{F}(\hat{M})]$ is a holomorphic multiplicative function without zeros on $\hat{V}$, with the multiplicator-set $\mu \circ \lambda$. Conversely, let ϕ be a holomorphic multiplicative function

without zeros on $\hat{V}$; then $\zeta = \frac{1}{2\pi i} d(\log \phi)$ is a holomorphic differential on V, hence of the form $\zeta = \theta - iC\theta$, with $\theta \in A'_0$; then we have

$$\phi(\hat{M}) = e\Big(\int_{\hat{P}}^{\hat{M}} \zeta + c\Big) = \psi[\hat{F}(\hat{M})],$$

with

$$\psi(x) = e[L(x) + c], \qquad L(x) = \langle \theta, x\rangle - i\langle C\theta, x\rangle = \langle \theta, x\rangle - i\langle \theta, Jx\rangle,$$

where c is a constant; as L is a complex-linear form on A, this shows that the multiplicator-set of ϕ is trivial.

If ψ is any meromorphic multiplicative function belonging to Θ, with the multiplicator-set μ, one can choose a constant $a \in A$ so that $\psi[\hat{F}(\hat{M}) + a]$ is not everywhere 0 or ∞, and then this is a meromorphic multiplicative function on V with the multiplicator-set $\mu \circ \lambda$; in conjunction with the results of no. 6, this shows that there exist meromorphic multiplicative functions on $\hat{V}$ with an arbitrarily given special multiplicator-set.

10. One can define the *divisor* on V of any meromorphic multiplicative function on $\hat{V}$, or more generally of any meromorphic function ϕ on $\hat{V}$ such that, for every $\sigma \in H$, $\phi(\sigma\hat{M})/\phi(\hat{M})$ is a holomorphic function without zeros on $\hat{V}$; the definition is briefly as follows (cf. K. Kodaira, [3], Part II). If $z_1, \cdots, z_n$ are local complex coordinates in a neighborhood of a point P of V, and $\hat{P}$ is a point lying over P on $\hat{V}$, $z_1, \cdots, z_n$ can be used as local coordinates on $\hat{V}$ at $\hat{P}$, and ϕ can be expressed as a product $\phi = E(z)\prod_j \phi_j(z)^{m_j}$ of a unit-factor $E(z)$ (i. e. a holomorphic function, $\neq 0$ at P) and of powers of irreducible holomorphic functions $\phi_j(z)$ (irreducible, that is, in the ring of holomorphic functions at $\hat{P}$). Replacing $\hat{P}$ by another point $\sigma\hat{P}$ of $\hat{V}$ lying over P will merely affect $E(z)$, but not the $\phi_j(z)$ or the m_j. The divisor of ϕ on V will then be defined locally, in a neighborhood of P, as $\sum_j m_j W_j$, where W_j is the irreducible algebroid variety, of complex dimension $n-1$, defined by $\phi_j(z) = 0$, with the orientation determined by the condition $\Omega^{n-1} > 0$. Then the divisor (ϕ) of ϕ on V will be defined globally, by means of any suitable finite open covering of V; it will be of the form $\sum_\rho a_\rho Z_\rho$, where the a_ρ are integers, and the Z_ρ are irreducible compact analytic (i. e., everywhere algebroid) subvarieties of V, of complex dimension $n-1$, with the orientation defined above. More generally, any such expression will be called a divisor (more precisely, an analytic divisor) on V.

11. Now, let again ϕ be a meromorphic multiplicative function on $\hat{V}$, and put $\zeta = \frac{1}{2\pi i} d(\log \phi)$; it follows from the multiplicative property of ϕ that this is a meromorphic differential form on V (more accurately, it is the inverse image on $\hat{V}$ of such a form). By analytic continuation, one sees that $\phi(\hat{M}) = e(\int_{\hat{P}}^{\hat{M}} \zeta + c)$, where c is a constant; the integral is taken along any differentiable path, from the fixed point $\hat{P}$ on $\hat{V}$ to $\hat{M}$, which does not meet (ϕ); hence, for $\sigma \in H$, we have $\phi(\sigma\hat{M}) = \phi(M) e(\int_{\sigma} \zeta)$. If ϕ is factored as $\phi = E(z) \prod_j \phi_j(z)^{m_j}$ in terms of local coordinates in the neighborhood of a point of $\hat{V}$, then $\zeta - \sum_j m_j \frac{1}{2\pi i} d(\log \phi_j)$ is holomorphic in that neighborhood; from this, one deduces, by first considering the case of a singular 2-cell in such a neighborhood and of its boundary, that, if S is any 2-dimensional singular chain in V, whose boundary C is a differentiable singular chain and does not meet (ϕ), the intersection-number of S and (ϕ) is equal to $\int_C \zeta$; in the language of the theory of currents (cf. [3]), this shows that the cycle (ϕ) of dimension $2n-2$ on V is the differential of the current ζ. This implies, firstly, that (ϕ) is homologous to 0 with real coefficients, hence also (since it is an integral cycle [4]) with rational coefficients. In particular, a holomorphic multiplicative function ϕ on $\hat{V}$ cannot have zeros: for otherwise the integral of Ω^{n-1} on (ϕ) would be > 0, and, as this is a closed form, (ϕ) could not be homologous to 0. Also, if $\tau \in T$, the linking coefficient of τ and (ϕ) is equal to $\int_{\tau} \zeta \bmod 1$ (cf. Igusa [7]); for, if t is any differentiable singular cycle of class τ which does not meet (ϕ), that linking coefficient is by definition the intersection-number of (ϕ) and of any 2-dimensional singular chain with real coefficients, with the boundary t, reduced mod 1. As it is known, by Poincaré's duality theorem, that the linking coefficient defines the torsion groups of V, of dimensions 1 and $2n-2$, as a dual pair, it follows that (ϕ) is homologous to 0 with integral coefficients if and only if $\int_{\tau} \zeta \equiv 0 \bmod 1$ for all $\tau \in T$, i. e. if and only if the multiplicator-set of ϕ is special.

12. Let now Z be any analytic divisor on V. We can cover V with open sets U_h, so small that, in each U_h, Z can be written as $\sum_j m_j W_j$, where

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[4] As to this and other "obvious" homological properties of the subvarieties of V and in particular of the divisors on V, they can best be justified by the definitions and results in N. Hamilton's forthcoming thesis [4].

W_j is the variety of zeros of an irreducible holomorphic function $\phi_j(z)$ in U_h, the z's being local complex coordinates in U_h. Define the differential form ζ_h in U_h by $\zeta_h = \sum_j m_j \frac{1}{2\pi i} d(\log \phi_j)$; then $\zeta_h - \zeta_k$ is holomorphic in $U_h \cap U_k$; hence, by the main theorem of [10], if Z is homologous to 0 with real coefficients (and only in that case), there will be a meromorphic differential form ζ on V, such that, in each U_h, $\zeta - \zeta_h$ is a closed holomorphic form. Analytic continuation and the monodromy principle then show that there exists a meromorphic multiplicative function ϕ on $\hat{V}$, such that, in each U_h, $\phi = E(z)\prod_j \phi_j(z)^{m_j}$, where $E(z)$ is a unit-factor in U_h, and that $\zeta = \frac{1}{2\pi i} d(\log \phi)$; then we have $(\phi) = Z$. By the results of no. 11, Z is homologous to 0 with integral coefficients if and only if the multiplicator-set of ϕ is special.

Now suppose that ϕ_1, ϕ are two meromorphic multiplicative functions such that $(\phi) = (\phi_1) = Z$; then ϕ_1/ϕ is a holomorphic multiplicative function without zeros, hence its multiplicator-set is trivial. Therefore, if $\mathfrak{G}_h$ is the group of the analytic divisors which are homologous to 0 with real coefficients, we have attached to each $Z \in \mathfrak{G}_h$ a coset $c(Z)$ of the group γ' of trivial multiplicator-sets in the group of all multiplicator-sets, consisting of the multiplicator-sets of all meromorphic multiplicative functions ϕ such that $(\phi) = Z$. This coset will consist of special multiplicator-sets, i. e. it will be in Γ'/γ', if and only if Z is in the subgroup $\mathfrak{G}_a$ of $\mathfrak{G}_h$ consisting of the divisors which are homologous to 0 with integral coefficients. We have $c(Z) = \gamma'$ if and only if Z is the divisor of a function ϕ with the multiplicator-set 1, i. e. of a meromorphic function on V; then we say that Z is linearly equivalent to 0; the group of such divisors will be denoted by $\mathfrak{G}_l$.

But Γ'/γ' can be canonically identified with Γ/γ, hence also with Γ_0 and Θ'. Therefore the mapping $Z \to c(Z)$, restricted to $\mathfrak{G}_a$, defines a canonical homomorphism of $\mathfrak{G}_a$ onto the Picard variety Θ' of V, with the kernel $\mathfrak{G}_l$; we may also say that we have defined a canonical isomorphism between $\mathfrak{G}_a/\mathfrak{G}_l$ and the character-group Γ_0 of Δ: this is Igusa's first duality theorem. Similarly, the mapping $Z \to c(Z)$ determines a canonical isomorphism between $\mathfrak{G}_h/\mathfrak{G}_l$ and a subgroup of finite index of the character-group of H; this subgroup will be the whole character-group of H whenever every character of H of finite order is the multiplicator-set of some meromorphic multiplicative function on $\hat{V}$, or, what amounts to the same in view of the above results, whenever every homology class of dimension $2n-2$ and of finite order on V (with integral coefficients) contains an analytic divisor. This is Igusa's second duality theorem. By a theorem of Lefschetz ([8]), V will have that property whenever it is a non-singular algebraic subvariety of a projective

space; indeed, it seems likely that, for such variety V, every covering manifold of V with a finite number of sheets is again an algebraic variety.

§ II. Construction of a system of representatives for the group of divisor-classes.

13. For the proof of the main result of this §, we shall need various lemmas on theta-functions. As explained in no. 10, if ϑ is a theta-function (other than 0) belonging to an abelian variety Θ, one can define the divisor (ϑ) of ϑ on Θ; this is a positive divisor, i. e. all its components have positive coefficients. If a point u of Θ lies on a component of (ϑ), we shall say that ϑ is 0 at u.

Periodicity conditions for theta-functions will always be understood to be in the normalized form given in no. 6. As this still depends upon the choice of a symmetric form E^0, when Θ and E are given, we shall further normalize this by choosing a set of generators $r_1, \cdots, r_{2q}$ for Δ, and taking $E^0(r_j, r_k) = E(r_j, r_k)$ for $1 \leqq j \leqq k \leqq 2q$; then E^0 depends linearly upon E. In applying this to a product $\Theta \times \Theta_1$ of two abelian varieties $\Theta = A/\Delta$, $\Theta_1 = A_1/\Theta_1$, it should be understood that, after having chosen the sets of generators for Δ, Δ_1, we take, as the generators for $\Delta \times \Delta_1$ those for Δ and those for Δ_1, in that order.

From this agreement it follows that, if E and E_1 are Riemann forms for Θ, and ϑ, ϑ_1 are theta-functions, belonging to Θ and E, and to Θ and E_1, respectively, then $\vartheta\vartheta_1$ is a theta-function belonging to $E + E_1$; and so is $\vartheta(x - E^{-1}E_1 a)\vartheta_1(x + a)$ for every $a \in A$, as one easily verifies. It is clear that one can choose a so that this is not 0 at a given point u of Θ, provided ϑ, ϑ_1 are not everywhere 0. In other words, if a Riemann form E is the sum of two Riemann forms, and if u is any point of Θ, there is a theta-function, belonging to Θ and E, which is not 0 at u.

14. If E is a Riemann form for Θ, the set of theta-functions belonging to Θ and E is a vector-space of finite dimension over $\boldsymbol{C}$. Let $(\vartheta_\mu)_{0 \leqq \mu \leqq M}$ be a basis for the space of theta-functions belonging to Θ and mE, where m is an integer > 0; by what has been said above, as soon as $m \geqq 2$, there is no point of Θ where all the ϑ_μ are 0; furthermore, by a theorem of Lefschetz which we have mentioned in no. 6, as soon as m has been taken large enough, the mutual ratios of the ϑ_μ cannot be the same at two points of A unless these points have the same image in Θ; let m be so chosen, once for all. Let now Z be an infinite subset of Θ; consider all the homogeneous polynomials

$\Phi(U_0, \cdots, U_M)$ in $M+1$ indeterminates, such that $\Phi(\vartheta_0, \cdots, \vartheta_M)$ is 0 on Z; they generate a homogenous ideal in the ring of all polynomials in $U_0, \cdots, U_M$, whose set of zeros, in the projective space of dimension M with the homogeneous coordinates $(u_0, \cdots, u_M)$, is infinite and therefore has at least one component of dimension > 0. In the vector-space of all homogeneous polynomials Φ of degree r in $U_0, \cdots, U_M$, consider the subspace of those for which $\Phi(\vartheta_0, \cdots, \vartheta_M)$ is 0 on Z; let $\delta(r)$ be its codimension, i. e. the difference between the dimensions of the space and of the subspace. By Hilbert's theorem on the characteristic function of a homogeneous ideal, $\delta(r)$ increases indefinitely with r.

Now take for Θ the dual Picard variety of a Hodge manifold V, and for Z the image $Z = F(V)$ of V by the canonical mapping F of V into Θ; Z is connected, hence it is an infinite set unless it is reduced to a point, in which case we must have $q = 0$; this trivial case will be excluded henceforward. The product $\vartheta\vartheta'$ of two theta-functions cannot be 0 on Z unless ϑ or ϑ' is 0 on Z; for, if $\vartheta\vartheta'$ is 0 on Z, $\vartheta[\hat{F}(\hat{M})]\vartheta'[\hat{F}(\hat{M})]$ is 0 on $\hat{V}$, and, as this is the product of two holomorphic functions on $\hat{V}$, one factor must be 0 on an open set, hence everywhere. The integer m and the set (ϑ_μ) having been chosen as above, we use Z, as above, to define a homogeneous polynomial ideal; and we choose r so that $\delta(r) \geqq q+1$.

15. Let E'' be the sum of two Riemann forms for the product $\Theta'' = \Theta \times \Theta'$ of Θ and of the dual variety Θ'. In our usual notation, we have $\Theta = A/\Delta$, $\Theta' = A'/\Delta'$, hence $\Theta'' = A''/\Delta''$ with $A'' = A \times A'$, $\Delta'' = \Delta \times \Delta'$; as usual, write A_0, A'_0, and $A''_0 = A_0 \times A'_0$ for the underlying real spaces of A, A', A''. As E'' is a bilinear form on $A''_0 \times A''_0$, it can be written, more explicitly, as $E''(x, x'; y, y')$, with $(x, x') \in A_0 \times A'_0$, $(y, y') \in A_0 \times A'_0$. Then, for each integer $N \geqq 0$, the bilinear form

$$E''_N(x, x'; y, y') = E''(x, x'; y, y') + NE(x, y)$$

is a Riemann form for Θ''; let Λ_N be the vector-space of all theta-functions belonging to Θ'' and E''_N. Take any $u' \in \Theta'$; we shall now prove that the codimension of the subspace of Λ_{mr}, consisting of the functions in it which are 0 on $Z \times u'$, is $\geqq q+1$.

In fact, as E'' is the sum of two Riemann forms, the result in no. 13 shows that we can choose a function ϑ'_0 in Λ_0 which is not 0 at (z, u'), z being an arbitrarily chosen point in Z. Now, Φ being a homogeneous polynomial of degree r in $U_0, \cdots, U_M$, put

$$\vartheta''_\Phi(x, x') = \vartheta'_0(x, x')\Phi[\vartheta_0(x), \cdots, \vartheta_M(x)];$$

the mapping $\Phi \to \vartheta''_\Phi$ is a linear mapping of the space of such polynomials into Λ_{mr}; and ϑ''_Φ is 0 on $Z \times u'$ if and only if $\Phi(\vartheta_0, \cdots, \vartheta_M)$ is 0 on Z. As the codimension of the inverse image of a linear subspace by a linear mapping is at most equal to that of the subspace, it follows that the subspace of Λ_{mr}, consisting of the functions in it which are 0 on $Z \times u'$, has at least the codimension $\delta(r)$.

16. Now we shall prove that there is a function in Λ_{mr} which is not 0 on any of the sets $Z \times u'$, where u' is any point of Θ'. In fact, let $(\vartheta''_\rho)_{0 \leqq \rho \leqq R}$ be a basis for Λ_{mr}; writing any function ϑ'' in Λ_{mr} as $\vartheta''_t = \sum_\rho t_\rho \vartheta''_\rho$, we can identify Λ_{mr} with the affine space $\boldsymbol{C}^{R+1}$ of dimension $R+1$ over $\boldsymbol{C}$; from this we can derive, in the usual manner, a projective space P^R of dimension R, with the homogeneous coordinates $(t_0, \cdots, t_R)$. The mutual ratios of the ϑ''_ρ are meromorphic functions on Θ'', hence algebraic functions on $\Theta'' = \Theta \times \Theta'$ when Θ, Θ' are identified with algebraic varieties by Lefschetz's theorem (cf. no. 6); therefore the condition that the function ϑ''_t should be 0 at a point (u, u') of Θ'' determines an algebraic set of points $((u, u'), (t_0, \cdots, t_R))$, i. e. a bunch of varieties, on the product-variety $\Theta'' \times P^R$. Hence, for any $z \in Z$, the condition that ϑ''_t should be 0 at (z, u') determines an algebraic set B_z in the product $\Theta' \times P^R$. Let B be the intersection of all the sets B_z, for all $z \in Z$; this is again an algebraic set; let B' be its projection on P^R. If $t = (t_0, \cdots, t_R)$ is not in B', then, to every $u' \in \Theta'$, there is a $z \in Z$ such that (u', t) is not in B_z, i. e. such that ϑ''_t is not 0 at (z, u'); hence ϑ''_t has the desired property. What we have to show is therefore that $B' \neq P^R$; and for this it is enough to show that no component of B can have a dimension $\geqq R$. Assume therefore that some component W of B has a dimension $\geqq R$; then, if u' is in the projection of W on Θ', the intersection $B \cap (u' \times P^R)$ has a component of dimension $\geqq R - q$. But this intersection is nothing else than the set of points (u', t) of $u' \times P^R$ such that ϑ''_t is 0 on $Z \times u'$; as such functions ϑ''_t form a subspace of Λ_{mr} of codimension $\geqq q+1$, it follows that $B \cap (u' \times P^R)$ is a linear subspace of $u' \times P^R$ of dimension $\leqq R - q - 1$, and this contradicts the above assumption.

17. After these preliminaries, we are now ready to attack the main problem of this §, which is to construct an algebraic family of divisors on V, parametrized by Θ', containing one and only one representative of each class in $\mathfrak{G}_a/\mathfrak{G}_1$.

In the first place, consider on $A''_0 \times A''_0$ the bilinear form

$$E''_0(x, x'; y, y') = \langle y', x \rangle - \langle x', y \rangle.$$

The dual of $A''_0 = A_0 \times A'_0$ being $A'_0 \times A_0$, this form can be written as $\langle E''_0 x'', y'' \rangle$, where $x'' = (x, x')$, $y'' = (y, y')$, and where E''_0 is the linear mapping $E''_0 = \begin{pmatrix} 0 & -I' \\ I & 0 \end{pmatrix}$ of $A_0 \times A'_0$ onto $A'_0 \times A_0$, I and $I' = {}^tI$ denoting again the identical automorphisms of A_0, A'_0. The scalar multiplication by i in $A'' = A \times A'$ induces on $A''_0 = A_0 \times A'_0$ the automorphism $J'' = \begin{pmatrix} J & 0 \\ 0 & J' \end{pmatrix}$. Then we have $E''_0(J''x'', y'') = \langle y', Jx \rangle + \langle x', Jy \rangle$; as this is symmetric, it follows that, as soon as the integers N, N' are large enough, the bilinear form

$$E''_{N,N'}(x'', y'') = E''_0(x'', y'') + N \cdot E(x, y) + N' \cdot E'(x', y')$$

is a Riemann form for $\Theta'' = \Theta \times \Theta'$, if E, E' are, as before, Riemann forms for Θ, Θ' respectively.

By no. 13, as soon as $N \geqq 2$, there is a theta-function ϑ, belonging to Θ and NE, which is not 0 on $Z = F(V)$. Also, $E(x, y) + E'(x', y')$ is a Riemann form for Θ''; hence, if $E''_{N,N'}$ is a Riemann form for Θ'', $E''_{N+1,N'+1}$ is the sum of two Riemann forms for Θ''; then the main result of no. 16, applied to this instead of E'', shows that there is a theta-function ϑ'', belonging to Θ'' and to $E''_{N+1+mr,N'+1}$, which is not 0 on any of the sets $Z \times u'$. If we write again N, N' instead of $N + 1 + mr$, $N' + 1$, we have thus shown that, for a suitable choice of N, N', there will be a theta-function ϑ, belonging to Θ and NE, which is not 0 on $F(V)$, and a theta-function ϑ'', belonging to Θ'' and $E''_{N,N'}$, which is not 0 on any set $F(V) \times u'$.

In this, it is understood that the periodicity conditions for these functions have been normalized as explained in no. 13; this implies that, if E^0, E'^0 are the symmetric forms attached, as there, to E, E', the symmetric form attached to $E''_{N,N'}$ is:

$$\langle y', x \rangle + \langle x', y \rangle + NE^0(x, y) + N'E'^0(x', y').$$

18. ϑ, ϑ'' being chosen as explained in no. 17, we now put

$$\eta(x, x') = \vartheta(x)\vartheta''(x, x')^{-1}.$$

This is a meromorphic function with the periodicity properties

$$\eta(x + r, x') = \eta(x, x')e(\tfrac{1}{2}\langle x', r \rangle + \tfrac{1}{2}\langle x', Jr \rangle),$$

$$\eta(x, x' + r') = \eta(x, x')e[L_{r'}(x, x')],$$

for $x \in A$, $x' \in A'$, $r \in \Delta$, $r' \in \Delta'$, $L_{r'}(x, x')$ being a certain complex-linear function on $A \times A'$, which depends upon r'. Hence, if we put, for any $a' \in A'$,

$$\psi_{a'}(x) = \eta(x, a')e(\tfrac{1}{2}\langle a', x\rangle - \tfrac{i}{2}\langle a', Jx\rangle),$$

we have

$$\psi_{a'}(x + r) = \psi_{a'}(x)e(\langle a', r\rangle),$$

$$\psi_{a'+r'}(x) = \psi_{a'}(x)e[L_{a',r'}(x)],$$

where $L_{a',r'}(x)$ is a complex-linear function on A, depending upon a', r'; $\psi_{a'}(x)$ is a meromorphic function on A, which, by the properties which we have assumed for ϑ and ϑ'', is not everywhere 0 or ∞ on $\hat{F}(\hat{V})$; and it is a multiplicative function, with the multiplicator-set $e(\langle a', r\rangle)$ of absolute value 1. This shows that $e[L_{a',r'}(x)]$ has the multiplicator-set 1, hence it is a constant (as could also readily have been verified by its explicit calculation). Therefore we have

$$\psi_{a'+r'}(x) = c(a', r')\psi_{a'}(x),$$

where $c(a', r')$ depends only upon a' and r'.

19. It follows from the relations in no. 18 that $\psi_{a'}[\hat{F}(\hat{M})]$ is a meromorphic multiplicative function, not everywhere 0 or ∞, on $\hat{V}$, with the multiplicator-set $\mu_{a'} \circ \lambda$, where $\mu_{a'}(r) = e(\langle a', r\rangle)$, and λ is as in no. 7. As $\psi_{a'}$ is multiplied by a constant when a' is replaced by $a' + r'$, its divisor $(\psi_{a'})$ on V depends only upon the image $u' = \omega(a')$ of a' by the canonical homomorphism ω of A' onto $\Theta' = A'/\Delta'$; hence we may write it as $(\psi_{a'}) = D(u')$.

$D(u')$ is also the divisor of the multiplicative function

$$\vartheta[\hat{F}(\hat{M})]\vartheta''[\hat{F}(\hat{M}), a']^{-1}$$

on $\hat{V}$. Now $\vartheta[\hat{F}(\hat{M})]$ is a holomorphic function on $\hat{V}$, which, when $\hat{M}$ is replaced by $\sigma\hat{M}$, is multiplied by an exponential factor; hence (cf. no. 10) one can define its divisor W_0 on V. Similarly one can define the divisor W of $\vartheta''[\hat{F}(\hat{M}), x']$ on $V \times \Theta'$. By purely local considerations, using the expression of this function as a product of irreducible holomorphic functions in the neighborhood of a point, one shows that the divisor of $\vartheta''[\hat{F}(\hat{M}), a']$ on V, for a fixed a', is the intersection of W with $V \times u'$, where $u' = \omega(a')$, each component of that intersection being counted with its multiplicity; more correctly it is the divisor $W(u')$ on V such that $W(u') \times u'$ is the intersection of W and $V \times u'$. There is, in a case such as this, no difficulty in defining the intersection-multiplicities, by function-theoretic or topological means.

Therefore we have $D(u') = W_0 - W(u')$; and what we have done shows that this is a divisor in the group $\mathfrak{G}_a$, which has the image u' in Θ' by the canonical homomorphism of $\mathfrak{G}_a$ onto Θ'. As Θ' is an algebraic variety, we may express the relation $W(u') \times u' = W \cdot (V \times u')$, where the $\cdot$ denotes the intersection, by saying that the $W(u')$ are an algebraic family of positive divisors, parametrized by the variety Θ'. The canonical homomorphism of $\mathfrak{G}_a$ onto Θ', with the kernel $\mathfrak{G}_l$, can then be defined by saying that it maps each divisor $X \in \mathfrak{G}_a$ onto the (uniquely determined) point u' of Θ' such that $W(u')$ is linearly equivalent to $W_0 - X$. This implies of course that $W(0)$ is linearly equivalent to W_0, so that, in this last statement, W_0 may be replaced by $W(0)$.

§ III. The main theorem.

20. By an analytic subset of a complex-analytic manifold, we understand a closed subset of it which, in some neighborhood of everyone of its points, is the set of common zeros of a finite number of holomorphic functions; an analytic subset which is irreducible (i. e. which is not the union of two other such sets) is called a subvariety. By a divisor X, we understand, as above, any formal sum of subvarieties of maximal dimension, with integral coefficients; if the latter are positive, X is said to be positive, and we write $X > 0$; we write $X > Y$ for $X - Y > 0$. By the carrier $|X|$ of X, we understand the union of the components of X.

In all this §, we shall denote by V a connected compact complex-analytic manifold of complex dimension n. If S is a complex-analytic manifold, and X a divisor on $V \times S$, the relation $X(s) \times s = X \cdot (V \times s)$ defines a divisor $X(s)$ on V for every $s \in S$ such that $V \times s$ is not contained in $|X|$; as $|X|$ is closed, the set S' of these points is an open subset of S, which may be considered as a complex-analytic manifold; we shall say that the set of all divisors $X(s)$, for $s \in S'$, is an *analytic family* of divisors on V, parametrized by S', or having S' as its parameter manifold.

Notations being as above, one may always assume that X has no component of the form $V \times T$, where T is a subvariety of S of maximal dimension; in fact, adding such a component to X does not modify $X(s)$ when s is not on T, and makes $X(s)$ undefined if s is on T. That being assumed, we shall show that the set $S - S'$ of points s of S for which $X(s)$ is not defined is an analytic subset of S, of (complex) dimension $\leqq p - 2$ if p is the dimension of S. It is clearly enough to prove this for $X > 0$. Each point $M \times s_0$ of $V \times S$ has a neighborhood where X can be written as

the divisor (Φ) of some holomorphic function Φ; let $z_1, \cdots, z_n$ be local coordinates on V in a neighborhood of M; then, if a point s, sufficiently near to s_0, is such that $V \times s \subset |X|$, we must have $\Phi(z, s) = 0$ identically in z near $z = 0$; therefore the points of $S - S'$ near s_0 are common zeros of all coefficients of Φ when Φ is expanded into a power-series in $z_1, \cdots, z_n$ with coefficients in the ring of holomorphic functions of s at $s = s_0$; conversely, if a point s, sufficiently near to s_0, is a common zero of all these coefficients, then it is easily seen, by analytic continuation along $V \times s$, that $V \times s$ is contained in $|X|$. Applying a theorem of H. Cartan,[5] one sees that $S - S'$ can be defined, in a sufficiently small neighborhood of s_0, by equating to 0 a finite number of the coefficients of Φ; therefore it is an analytic set. If now T is any component of $S - S'$, $V \times T$ is contained in $|X|$; therefore, if T were of dimension $p - 1$, $V \times T$ would be a component of X, which is against our assumption.

In particular, if S is connected, S' also is connected. Then the homological theory of subvarieties of a complex-analytic manifold ([4]) shows that any two divisors of the form $X(s)$ are homologous to each other on V; in particular, if $X(s)$ is homologous to 0 for some s, the same is true for all $s \in S'$.

The main purpose of this § is to show that, *if V is a Hodge manifold, the canonical mapping c of the group $\mathfrak{G}_a$ of the divisors on V, homologous to 0 on V, into the Picard variety Θ' of V* (§ I, no. 12) *induces an analytic mapping into Θ' of every analytic family of divisors belonging to $\mathfrak{G}_a$ on V.* In other words, if S, X, S' and $X(s)$ are as above, and if $X(s)$ is homologous to 0 for all $s \in S'$, the mapping $s \to c[X(s)]$ is a complex-analytic mapping of S' into Θ'. Assuming for a moment that this is so, we shall show that *this mapping can be extended to a complex-analytic mapping of S into Θ'.* In fact, write as before $\Theta' = A'/\Delta'$, where A' is a complex vector-space and Δ' a discrete subgroup of A'; if x' is a variable point of A', we may consider dx' as a vector-valued differential form on A' (with values in A'); as this form is invariant by translations in A' and in particular by Δ', it induces on Θ' a vector-valued form which, in terms of the complex structure of Θ', is a holomorphic form; its inverse image ω on S' by the mapping $s \to c[X(s)]$ is therefore a holomorphic form on S'. As $S - S'$ is an analytic subset of S and has no component of dimension $p - 1$, it follows, by Hartogs' theorem, that ω can be extended to a holomorphic form on S. As every 1-dimensional cycle on S can be deformed into one on S', the periods of ω on S are the

[5] "Théorème a" of [1], p. 191.

same as those on S'; as the latter must belong to Δ', the same is therefore true on S; as in § I, one concludes from this that the mapping $\hat{s} \to \int_{\hat{a}}^{\hat{s}} \omega$ of the universal covering of S into A' (for a fixed $\hat{a}$) induces a holomorphic mapping of S into Θ', which, on S', coincides (up to a translation) with the given one $s \to c[X(s)]$. This proves our assertion.

Even so, our result will remain incomplete in one important respect: we shall not prove it for families of divisors which are parametrized by algebroid varieties. By an algebroid variety, we understand one on which each point has a neighborhood isomorphic to a locally irreducible analytic subset of a complex-analytic manifold. It seems likely that the proof which is to be given here of our main theorem could be applied to this more general problem with no more than slight modifications, or at any rate that the more general result could be shown to follow from ours, if only the foundations of algebroid geometry were not even now as shaky as those of algebraic geometry once used to be.[6]

21. Let $X(s)$ and $Y(t)$ be two analytic families of divisors on V, respectively parametrized by the manifolds S and T. We shall now prove that *the set of points* (s, t) *of* $S \times T$ *such that* $X(s) \succ Y(t)$ *is an analytic subset of* $S \times T$. If we put $\bar{X} = X \times T$, $\bar{X}$ is a divisor on $V \times (S \times T)$, and we have $\bar{X}(s, t) = X(s)$; operating similarly on Y, we see that it is enough to show that, *if* $X(s)$, $Y(s)$ *are two families, parametrized by the same manifold* S, *the set of points* $s \in S$ *such that* $X(s) \succ Y(s)$ *is an analytic subset of* S. If we write $X = X' - X''$, $Y = Y' - Y''$, where X', X'', Y', Y'' are positive divisors, $X(s) \succ Y(s)$ is equivalent to $X'(s) + Y''(s) \succ X''(s) + Y'(s)$; therefore it will be enough to prove our assertion for positive X and Y. As the question is purely local so far as S is concerned, we may assume that S is a neighborhood of 0 in the space of p complex variables.

Let M be any point on V; take local coordinates $(w, z_1, \cdots, z_{n-1})$ on V in a neighborhood of M, such that the subvariety (of complex dimension 1) of that neighborhood defined by $z_1 = \cdots = z_{n-1} = 0$ is not contained in $|X(0)|$ nor in $|Y(0)|$. Then, by Weierstrass's lemma, X can be written in a neighborhood of $M \times 0$ on $V \times S$ as the divisor (Φ) of a function Φ of the form

[6] It is to be hoped that someone will soon undertake the taxing but necessary task of consolidating them or rather building them up anew. As some at least of the main difficulties have been removed by the recent work of H. Cartan, W. L. Chow, N. Hamilton, K. Oka and others, the time seems ripe for such an undertaking.

$$\Phi(w, z, s) = w^a + \sum_{i=1}^{a} w^{a-i}\phi_i(z, s),$$

where the ϕ_i are holomorphic and 0 at the origin; similarly, Y can be written as (Ψ) in a neighborhood of $M \times 0$, with Ψ of the form

$$\Psi(w, z, s) = w^b + \sum_{j=1}^{b} w^{b-j}\psi_j(z, s),$$

where the ψ_j are holomorphic and 0 at the origin. Considering Φ and Ψ as polynomials over the ring of holomorphic functions of z, s at the origin, we can divide Φ by Ψ; let R be the remainder, which is of the form $R(w, z, s) = \sum_{h=0}^{b-1} w^h \chi_h(z, s)$, where the χ_h are holomorphic at the origin. Put $|z| = \sup_{1 \leqq i \leqq n-1} |z_i|$; take $\rho > 0$ and an open neighborhood S' of 0 on S so that the range of the local coordinates (w, z) contains the closure of the polycylinder P defined by $|w| < \rho$, $|z| < \rho$, that Φ, Ψ are holomorphic in $P \times S'$, and that X, Y coincide respectively with (Φ) and with (Ψ) in $P \times S'$. Take $\epsilon > 0$ and $< \rho$ and a neighborhood S'' of 0 in S' so that, for $|z| < \epsilon$ and $s \in S''$, Φ and Ψ, considered as polynomials in w, have all their roots inside the disk $|w| < \rho$. Then, if $\zeta = (\zeta_1, \cdots, \zeta_{n-1})$ is such that $|\zeta| < \epsilon$, and if $s \in S''$, the roots of $\Phi(w, \zeta, s) = 0$, counted with their multiplicities, make up the intersection of $X(s)$ with the subvariety of P (of complex dimension 1) defined by $z = \zeta$; and the roots of $\Psi(w, \zeta, s) = 0$ make up the intersection of Y with that subvariety; therefore, if $X(s) \succ Y(s)$ for some $s \in S''$, $\Phi(w, \zeta, s)$ must be a multiple of $\Psi(w, \zeta, s)$, i. e. $R(w, \zeta, s)$ must be 0, whenever $|\zeta| < \epsilon$; for such an s, each coefficient $\chi_h(z, s)$ of R must therefore be identically 0 as a function of z; if the $\chi_h(z, s)$ are expanded into power-series in $z_1, \cdots, z_{n-1}$ with coefficients which are holomorphic functions of s in S', every $s \in S''$ such that $X(s) \succ Y(s)$ must be a common zero of all these coefficients. Conversely, if s is such a common zero, then $R = 0$, and so Φ is a multiple of Ψ in the ring of polynomials over holomorphic functions of z in $|z| < \rho$ and a fortiori in the ring of holomorphic functions in P. Therefore, when that is so, no component of $X(s) - Y(s)$ which has a coefficient < 0 in that divisor can have any point in P.

To every point M on V, we have thus assigned a neighborhood $P = P(M)$ of M with the above stated properties; V can then be covered with a finite number of such neighborhoods $P_\nu = P(M_\nu)$. To each M_ν, the above construction also assigns a neighborhood S''_ν of 0 on S; we may replace the S''_ν by their intersection S''. Then our construction defines infinitely many

functions $f_\alpha(s)$, holomorphic in S'', viz. the coefficients of the expansions into power-series in z of the functions $\chi_h(z, s)$ attached to the various points M_ν, with the following properties: if $s \in S''$ is such that $X(s) \succ Y(s)$, then all the $f_\alpha(s)$ are 0; conversely, if $s \in S''$ and all the $f_\alpha(s)$ are 0, then no component of $X(s) - Y(s)$ with a coefficient < 0 in that divisor can have a point in any P_ν; as the P_ν are a covering of V, this implies that $X(s) \succ Y(s)$. By Cartan's theorem,[5] this completes our proof.

If $X(s)$, $Y(t)$ are again two analytic families of divisors on V, then *the set of points* (s, t) *of* $S \times T$ *such that* $X(s) = Y(t)$ *is an analytic subset of* $S \times T$; in fact, it is the intersection of the two analytic subsets determined by $X(s) \succ Y(t)$ and by $Y(t) \succ X(s)$.

22. If X is any divisor on V, we shall denote by $L(X)$ the set of all meromorphic functions ϕ on V which are either identically 0 or such that $(\phi) \succ -X$; this is a vector-space over the field of constants. We shall now prove that, *for every divisor* X *on* V, *the vector-space* $L(X)$ *is of finite dimension.*

If $X = X' - X''$, with positive X' and X'', we have $L(X) \subset L(X')$; so it will be enough to prove our theorem for a positive divisor X. Then every point M on V has a neighborhood $U = U(M)$ in which X can be written as the divisor (Φ_M) of a function Φ_M, holomorphic in U; and the set $\Phi_M \cdot L(X)$ of all functions $\psi = \Phi_M \phi$, with $\phi \in L(X)$, consists of functions which are holomorphic in U. Consider the ideal generated by $\Phi_M L(X)$ in the ring of holomorphic functions at M; by Cartan's theorem,[5] this ideal is generated by a finite number of elements $\psi_i^{(M)}$ of $\Phi_M L(X)$, and there is an open neighborhood $U' = U'(M)$ of M and a constant $k = k(M)$ with the following properties: the closure $\bar{U}'$ of U' is contained in U; and every $\psi \in \Phi_M L(X)$ such that $|\psi| \leqq 1$ in $\bar{U}$ can be expressed as $\psi = \sum_i \lambda_i \psi_i^{(M)}$, where the λ_i are holomorphic and such that $|\lambda_i| \leqq k$ on $\bar{U}'$. Take a covering of V by a finite number of neighborhoods $U'_\nu = U'(M_\nu)$; to each there belongs a function $\Phi_\nu = \Phi_{M_\nu}$, a constant $k_\nu = k(M_\nu)$, and a finite set of functions $\psi_i^{(M_\nu)} \in \Phi_\nu L(X)$; put $k = \sup_\nu k_\nu$, and call $\phi_1, \cdots, \phi_m$ the functions $\Phi_\nu^{-1} \psi_i^{(M_\nu)}$; these are functions in $L(X)$. Put $C_\nu = \bar{U}'_\nu$. Then, for every $\phi \in L(X)$ and every ν, there are functions $\lambda_{\nu j}$ holomorphic on C_ν, such that $\phi = \sum_{j=1}^{m} \lambda_{\nu j} \phi_j$ on C_ν; furthermore, if $|\Phi_\nu \phi| \leqq 1$, on C_ν, the $\lambda_{\nu j}$ may be taken such that $|\lambda_{\nu j}| \leqq k$ on C_ν.

Now put

$$\| \phi \| = \sup_\nu \sup_{C_\nu} |\Phi_\nu \phi|$$

for each $\phi \in L(X)$; then every $\phi \in L(X)$ can be written in C_ν as $\phi = \sum_j \lambda_{\nu j}\phi_j$, with $|\lambda_{\nu j}| \leqq k \| \phi \|$ on C_ν. We shall now prove that, if the vector-space $L(X)$ is considered as a normed vector-space with the norm $\| \phi \|$, it is locally compact, i. e. that the "unit-sphere" $\| \phi \| \leqq 1$ is compact. In fact, take any sequence ϕ_α of functions in $L(X)$ such that $\| \phi_\alpha \| \leqq 1$; in C_ν, ϕ_α can be written as $\phi_\alpha = \sum_j \lambda_{\nu j}^{(\alpha)}\phi_j$, with $|\lambda_{\nu j}^{(\alpha)}| \leqq k$ on C_ν. By the well-known elementary theorem on sequences of bounded holomorphic functions, we may, after replacing the sequence ϕ_α by a suitable subsequence, assume that, for each ν and each j, the sequence $\lambda_{\nu j}^{(\alpha)}$ converges to a limit $\lambda_{\nu j}$ uniformly on every closed subset of U'_ν. Then one sees immediately that there is a function $\phi \in L(X)$ such that $\phi = \sum \lambda_{\nu j}\phi_j$ in U'_ν for each ν; furthermore, as we have

$$\Phi_\nu\phi = \sum_j \lambda_{\nu j}(\Phi_\nu\phi_j) = \lim_{\alpha\to\infty} \sum_j \lambda_{\nu j}^{(\alpha)}\Phi_\nu\phi_j = \lim_{\alpha\to+\infty} (\Phi_\nu\phi_\alpha)$$

uniformly on every closed subset of U'_ν, we see that the sequence ϕ_α tends to ϕ in the topology defined by the norm $\| \phi \|$, and that $\| \phi \| \leqq 1$. This shows that the normed vector-space $L(X)$ is locally compact; by a well-known elementary theorem of Banach, this implies that it has a finite dimension.

23. From now on, we assume that V is a Hodge manifold; and we proceed to prove the theorem stated in no. 20. Let $X(s)$ be an analytic family of divisors, parametrized by a manifold S; we have to prove that, if the divisors $X(s)$ are homologous to 0, $s \to c[X(s)]$ is a complex-analytic mapping of S into Θ'. We may assume that S is connected; and we choose a point s_0 on S. Call X', $-X''$ the positive and negative parts of X, i. e. the sums of terms, with positive and negative coefficients respectively, in the expression of X in terms of its irreducible components. Then $X(s)$ is the sum of the divisors $X'(s) - X'(s_0)$, $-[X''(s) - X''(s_0)]$ and $X(s_0)$, which are all homologous to 0, and it is enough to prove our assertion for the families $X'(s) - X'(s_0)$, $X''(s) - X''(s_0)$. In other words, it will be enough to show that if $X(s)$ is any family of positive divisors parametrized by the connected manifold S, $s \to c[X(s) - X(s_0)]$ is an analytic mapping of S into Θ'. Now, in § II, we have constructed an analytic family $W(u')$ of positive divisors on V, parametrized by Θ', such that $u' = c[W(0) - W(u')]$. This implies that there is one and only one point u' on Θ', given by $u' = c[X(s) - X(s_0)]$, for which $W(0) - W(u')$ is linearly equivalent to $X(s) - X(s_0)$, i. e. $W(u') + X(s)$ to $W(0) + X(s_0)$. Let $\phi_0, \cdots, \phi_m$ be a basis for the vector-

space $L[W(0) + X(s_0)]$; then, if s and u' are as we have said, there is one and only one point ξ in the projective m-dimensional space P^m such that, if $(\xi_0, \cdots, \xi_m)$ is a set of homogeneous coordinates for ξ, the divisor of the function $\phi = \sum_{i=0}^{m} \xi_i \phi_i$ is

$$(\phi) = W(u') + X(s) - [W(0) + X(s_0)].$$

For every $\phi = \sum_i \xi_i \phi_i$ in $L[W(0) + X(s_0)]$, put $Z(\xi) = (\phi) + W(0) + X(s_0)$, where ξ denotes the point with the homogeneous coordinates $(\xi_0, \cdots, \xi_m)$ in P^m; one sees at once that $Z(\xi)$ is an analytic family of divisors parametrized by P^m. By the final result of no. 21, the set of all points (s, u', ξ) in $S \times \Theta' \times P^m$ such that $Z(\xi) = W(u') + X(s)$ is an analytic subset Γ of $S \times \Theta' \times P^m$. But we have just shown that, for every $s \in S$, there is one and only one point (u', ξ) in $\Theta' \times P^m$ such that (s, u', ξ) is in Γ; as Γ is closed and Θ' and P^m are compact, this implies in the first place that Γ is the graph of a continuous mapping f of S into $\Theta' \times P^m$; it is then easily seen that f induces a holomorphic mapping on every subvariety of complex dimension 1 of any open subset of S; from this, using elementary results on functions of several complex variables, one deduces easily that f is holomorphic on S itself. This completes the proof.

The same method can be applied to the determination of all positive divisors in a given homology class. Consider a homology class which contains at least one positive divisor X_0; let $\phi_0, \cdots, \phi_m$ be a basis for the vector-space $L[W(0) + X_0]$; for every $\phi = \sum_i \xi_i \phi_i$ in that space, put $Z(\xi) = (\phi) + W(0) + X_0$, ξ being the point in P^m with the homogeneous coordinates $(\xi_0, \cdots, \xi_m)$. Then we see, as above, that for every positive divisor X homologous to X_0 there is one and only one point (u', ξ) of $\Theta' \times P^m$ such that $Z(\xi) = W(u') + X$; therefore this formula determines a one-to-one correspondence between such divisors X and the points (u', ξ) of $\Theta' \times P^m$ such that $Z(\xi) \succ W(u')$. As we have shown, the latter relation determines an analytic subset of $\Theta' \times P^m$, which, since Θ' and P^m are compact, is the union of a finite number of subvarieties of $\Theta' \times P^m$; moreover, as Θ' is an algebraic variety, these are algebraic subvarieties of $\Theta' \times P^m$ by Chow's theorem ([2]). This shows that *the positive divisors in a given homology class on V make up a finite number of algebraic families*, i. e. of families parametrized by algebraic varieties; of course the latter may have singular points and so need not be complex-analytic manifolds, but are algebroid varieties in the sense defined at the end of no. 20. One may also observe that for a given u' in Θ' the points ξ of P^m

such that $Z(\xi) \succ W(u')$ are those in a linear subvariety of P^m, corresponding to the subspace $L[W(0) + X_0 - W(u')]$ of the vector-space $L[W(0) + X_0]$; in particular, each one of the above families of positive divisors may be considered as a fibre-variety over a subvariety of Θ', the fibres being projective spaces.

§ IV. The higher Jacobian varieties.

24. I shall take this opportunity for making explicit some results implicitly contained in Hodge's book ([5]; cf. also [6]), as this can be done very simply in the language which has been introduced above in § I and throws some additional light on the results of that §.

As before, let V be a Kähler manifold of complex dimension n; take p such that $0 \leqq p \leqq n-1$; call now A_0 and A'_0 the homology groups of V over the real number-field $\boldsymbol{R}$ for the dimensions $2p+1$ and $2n-2p-1$ respectively. These are vector-spaces of finite dimension over $\boldsymbol{R}$; call Δ and Δ' the subgroups of A_0 and of A'_0 consisting of those homology classes which contain cycles with integral coefficients (i. e. the images in A_0 and in A'_0 of the homology groups of V with integral coefficients for the same dimensions); Δ is a discrete subgroup of A_0 of maximal rank, i. e. of rank equal to the dimension of A_0, so that A_0/Δ is a torus of that same dimension; and the same is true of A'_0 and Δ'.

By the Poincaré duality theorem, A_0 and A'_0 are put into duality with each other by the intersection-number (or " Kronecker index "), which induces a bilinear form $\langle x', x\rangle$ on $A'_0 \times A_0$; moreover, by the same theorem, Δ and Δ' are associated to each other in that duality, i. e. Δ' consists of all the elements r' of A'_0 such that $\langle r', r\rangle \equiv 0 \bmod 1$ for all $r \in \Delta$. We may therefore identify A'_0 with the dual space to A_0, i. e. with the cohomology group of V of dimension $2p+1$ with real coefficients, and similarly A_0 may be identified with the cohomology group of V of dimension $2n-2p-1$ with real coefficients. In particular, A_0 and A'_0 may be identified with the de Rham groups of degrees $2n-2p-1$ and $2p+1$ respectively; by the de Rham group of degree d, we understand the group of closed differential forms of degree d on V modulo the exterior differentials of forms of degree $d-1$. Then Δ, Δ' are the classes of forms of degrees $2n-2p-1$ and $2p+1$ with integral periods; and if elements x, x' of A_0, A'_0 are the cohomology classes of two closed forms ω, ω' of respective degrees $2n-2p-1$ and $2p+1$, we have $\langle x', x\rangle = \int_V \omega'\omega$.

By Hodge's existence theorems, each cohomology class contains one and only one harmonic form; so A_0 may also be identified with the space of harmonic forms of degree $2n-2p-1$. As in no. 7, consider the operator C derived from the complex structure of V; if ω is a form of degree d, we have $C(C\omega) = (-1)^d\omega$, so that C induces an operator such that $C^2 = -I$ on forms of any odd degree. As C commutes with the operator $\Delta = d\delta + \delta d$, it transforms harmonic forms into harmonic forms; call J the operator induced by $-C$ on the space A_0 of harmonic forms of degree $2n-2p-1$; this is an endomorphism of A_0 satisfying $J^2 = -I$; therefore A_0 must be of even dimension, and J can be used to define a complex structure on A_0; with that structure, A_0 becomes a vector-space over the field of complex numbers, which will be denoted by A. In a similar manner, A'_0 becomes a vector-space A' over complex numbers by means of the operator J' induced by $-C$ on harmonic forms of degree $2p+1$. Let ω, ω' be two harmonic forms of respective degrees $2n-2p-1$, $2p+1$; as C is an automorphism of the ring of differential forms, we have $C(\omega'\omega) = C\omega' \cdot C\omega$; as every form ζ of degree $2n$ can be written as $f dz_1 d\bar{z}_1 \cdot \cdots \cdot dz_n d\bar{z}_n$ in terms of local coordinates, we have $C\zeta = \zeta$ for such a form, and in particular $C(\omega'\omega) = \omega'\omega$. This gives $J'\omega' \cdot J\omega = \omega'\omega$, whence $\langle J'\omega', J\omega\rangle = \langle \omega', \omega\rangle$, or, if we replace ω by $J^{-1}\omega$, $\langle J'\omega', \omega\rangle = \langle \omega', J^{-1}\omega\rangle$; this shows that J' is the transpose of J^{-1} for the duality determined by the bilinear form $\langle x', x\rangle$. According to our definitions in § I, no. 1, this makes A' the "antidual" space of A; and, if we put $\Theta = A/\Delta$, $\Theta' = A'/\Delta'$, these are two complex toruses, dual to each other.

25. We shall now give a definition of J, as an endomorphism of the de Rham group of degree $2n-2p-1$, which depends only upon the complex structure of V and not upon the choice of a Kähler metric; the existence of a Kähler metric on V, however, is essential to our purposes. Take any d such that $0 \leqq d \leqq 2n$; in each cohomology class of degree d, there are forms ω such that $d\omega = d(C\omega) = 0$; in fact, the harmonic form of that class has that property. We now prove that, for $d\omega = 0$ and $d(C\omega) = 0$, the homology class of $C\omega$ depends only upon that of ω. In fact, assume that ω is homologous to 0; let ζ be any harmonic form of degree $2n-d$, and put $\zeta' = C^{-1}\zeta$; then ζ' is harmonic; we have $\zeta'\omega = C(\zeta'\omega)$ since $\zeta'\omega$ is of degree $2n$, and $C(\zeta'\omega) = C\zeta' \cdot C\omega = \zeta(C\omega)$, and so $\zeta'\omega = \zeta(C\omega)$. As ω is homologous to 0, we have $\int_V \zeta'\omega = 0$; this gives $\int_V \zeta(C\omega) = 0$; as this is true for all harmonic forms ζ, i. e. for representatives of all cohomology classes of degree

$2n-d$, and as $C\omega$ is closed, it implies that $C\omega$ is homologous to 0.[7] If now x is a cohomology class of degree d, and ω is a representative of that class such that $d\omega = 0$ and $d(C\omega) = 0$, we shall denote by Cx the class of the form $C\omega$; this is an endomorphism of the de Rham group of dimension d which satisfies $C^2 = (-1)^d I$ and depends only upon the complex structure of V; for $d = 2n-2p-1$, the operator J defined above is $J = -C$.

The complex torus Θ defined above by means of the cohomology group A_0 of degree $2n-2p-1$, of the complex structure determined by J, and of the subgroup Δ of classes of A_0 with integral periods, will henceforth be denoted by Θ_p; with this notation, the torus denoted above by Θ' now appears as Θ_{n-p-1}. Instead of A, Δ, we now write $A^{(p)}$, $\Delta^{(p)}$, so that $\Theta_p = A^{(p)}/\Delta^{(p)}$.

Thus, if V is any connected compact complex-analytic manifold of complex dimension n on which there exists at least one Kähler metric, we have defined a sequence $\Theta_0, \Theta_1, \cdots, \Theta_{n-1}$ of complex toruses invariantly attached to it, in such a way that Θ_p and Θ_{n-p-1} are dual to each other for $0 \leqq p \leqq n-1$. The torus Θ attached to V in § I, no. 7, is no other than Θ_0 in this notation, and its dual Θ' is Θ_{n-1}.

26. It will now be shown that, if V is a Hodge manifold, not only Θ_0 and Θ_{n-1} (as we have seen in § I) but all the Θ_p are abelian varieties; these will be called *the Jacobian varieties of* V. As before, denote by Ω the fundamental form of degree 2 of the Kähler metric, which by assumption has integral periods; we have $C\Omega = \Omega$. If x is the cohomology class of a form ω, that of $\Omega\omega$ will be denoted by Lx. If ω is a representative of the class x such that $d\omega = 0$ and $d(C\omega) = 0$, we have $d(\Omega\omega) = 0$ and $d[C(\Omega\omega)] = d[\Omega(C\omega)] = 0$, and therefore, according to our definitions, $L(Cx) = C(Lx)$. For forms of odd degree, this gives $L(Jx) = J(Lx)$; so L is a complex-linear mapping of $A^{(p)}$ into $A^{(p-1)}$ for all p. As $\Omega\omega$ has integral periods whenever ω has integral periods, L maps $\Delta^{(p)}$ into $\Delta^{(p-1)}$; so L induces a complex-analytic homomorphism of Θ_p into Θ_{p-1}, which will again be denoted by L. But by

[7] Using de Rham's operator H (cf. [3], § 19), this can be proved more briefly: if $d\omega = 0$, $H\omega = 0$ is a necessary and sufficient condition for ω to be homologous to 0; as H commutes with C, $H\omega = 0$ implies $H(C\omega) = 0$. This argument applies also to the projection operators $P_{a,b}$ which map each form ω of degree $a+b$ into the sum of the terms of type (a,b) in the expression of ω in terms of local complex coordinates. As $d\omega = d(C\omega) = 0$ implies $d(P_{a,b}\omega) = 0$, and as H commutes with $P_{a,b}$, one can define a projection operator $P_{a,b}$, invariantly related to the complex structure on V, on the cohomology group of V with real coefficients for the dimension $a+b$. The dimension of the image of this group under $P_{a,b}$ is Hodge's invariant $\rho^{a,b}$ ([6], Theorem I, p. 109).

Hodge's theorems, if ω is harmonic, so is $\Omega\omega$, and the mapping $\omega \to \Omega\omega$ of the space of harmonic forms of degree d into the space of harmonic forms of degree $d+2$ has the kernel 0 if $d \leqq n-1$ and maps the former space onto the latter if $d \geqq n-1$. So L maps Θ_p onto Θ_{p-1} for $2p \leqq n$, and it maps Θ_p into Θ_{p-1} with a finite kernel if $2p \geqq n$.

For $d \leqq n-1$, call B, B_1, B_2 the spaces of harmonic forms of degrees d, $d+2$, $2n-d$ respectively. As we have just said, L maps B isomorphically onto its image $L(B)$ in B_1. On the other hand, call E the kernel of the mapping L^{n-1-d} of B_1 into B_2, i. e. of the mapping $\omega \to \Omega^{n-1-d}\omega$; E is the space of "effective" harmonic forms of degree $d+2$ in Hodge's terminology, and Hodge has proved that B_1 is the direct sum of $L(B)$ and of E. Let Δ, Δ_1, Δ_2 be the groups of forms with integral periods in B, B_1, B_2 respectively; as we have said, L maps Δ into Δ_1, and so L^{n-1-d} maps Δ_1 into Δ_2; taking sets of generators of Δ, Δ_1, Δ_2 as bases for B, B_1, B_2 respectively, we see that the mappings L and L^{n-1-d} will then be given by matrices with integral coefficients, and so the linear subspaces $L(B)$ and E of B_1 can both be defined by linear equations with integral coefficients. This implies that the group $\Delta'_1 = L(\Delta) + (E \cap \Delta_1)$ is a subgroup of B_1 of maximal rank and a subgroup of Δ_1 of finite index, and that $E \cap \Delta_1$ is a subgroup of E of maximal rank.

Now, assuming that d is odd, we give to B, B_1, B_2 the complex structure determined by $J = -U$; as L and L^{n-1-d} are complex-linear mappings, $L(B)$ and E are then complex-linear subspaces of B_1. If we put $d = 2p-1$, we have $2p \leqq n$, and, with the notation explained above, $B = A^{(n-p)}$, $B_1 = A^{(n-p-1)}$, $\Delta = \Delta^{(n-p)}$, $\Delta_1 = \Delta^{(n-p-1)}$, and therefore $\Theta_{n-p} = B/\Delta$, $\Theta_{n-p-1} = B_1/\Delta_1$. Put $\Theta'_{n-p-1} = B_1/\Delta'_1$; the identical mapping of B_1 onto itself maps Δ'_1 into Δ_1, and so it induces a complex homomorphism of Θ'_{n-p-1} onto Θ_{n-p-1}; therefore these two toruses are isogenous (§ I, no. 2); but the former is the direct product of the complex torus $L(B)/L(\Delta)$, which is isomorphic to $B/\Delta = \Theta_{n-p}$, and of the complex torus $\Theta''_{n-p-1} = E/(E \cap \Delta_1)$. So Θ_{n-p-1} is isogenous to the product of Θ_{n-p} and of Θ''_{n-p-1} whenever $2p \leqq n$; and if we prove that Θ''_{n-p-1} is an abelian variety for $2p \leqq n-1$, it will follow by induction on p that the same is true of Θ_{n-p-1} for the same values of p, and also of Θ_p for these values of p since Θ_p is dual to Θ_{n-p-1}; as every integer q such that $0 \leqq q \leqq n-1$ can be written either as $q = p$ or as $q = n-p-1$ with $2p \leqq n-1$, our proof will then be complete. Now consider the bilinear form

$$F(\omega, \omega') = (-1)^{p+1} \int_V \omega\omega'\Omega^{n-2p-1}$$

for $\omega \in E$, $\omega' \in E$; as ω, ω' are of odd degree, we have $\omega\omega' = -\omega'\omega$, and F is alternating. If ω, ω' are in $E \cap \Delta_1$, i. e. if they have integral periods, $F(\omega, \omega')$ is an integer. Finally, again by Hodge's theorems, we have, for every $\omega \in E$:

$$\Omega^{n-2p-1}(C\omega) = (-1)^{p+1}(n-2p-1)!\,{}^*\omega,$$

and therefore

$$F(J\omega, \omega') = (n-2p-1)! \int_V \omega'({}^*\omega);$$

this is symmetric and positive-definite; so F is a Riemann form for Θ''_{n-p-1}, and this torus is an abelian variety.

27. By an analytic cycle of (complex) dimension p on V, we understand a finite sum, with integral coefficients, of subvarieties of V of complex dimension p. Let $\mathfrak{G}_h^{(p)}$ be the group of those cycles of dimension p on V which are homologous to 0 on V (with integral coefficients). Under the assumption that each analytic cycle on V defines a "current," i. e. essentially that the integral of a differential form of degree $2p$ on the manifold of simple points of a subvariety of V of complex dimension p is always convergent, one can define a canonical mapping of the group $\mathfrak{G}_h^{(p)}$ into the complex torus Θ_p. This assumption is "undoubtedly" true, but no proof of it seems to have been published; in order to simplify the definition of the canonical mapping of $\mathfrak{G}_h^{(p)}$ into Θ_p, I shall assume even more, viz. that every cycle in $\mathfrak{G}_h^{(p)}$ can be expressed as the boundary of a singular chain of such a nature that differential forms of degree $2p+1$ can be integrated on it.

That being assumed, let X be any cycle in $\mathfrak{G}_h^{(p)}$; put $X = bQ$, where b is the boundary operator and Q is a chain with integral coefficients. Let x' be a cohomology class of degree $2p+1$ with real coefficients; take a representative ω for x' such that $d\omega = 0$ and $d(C\omega) = 0$; we shall prove that $\int_Q \omega$ depends only upon x'. In fact, assume that ω is homologous to 0. As ω is then orthogonal to all harmonic forms, there is a form ζ such that $\omega = \Delta\zeta = d\delta\zeta + \delta d\zeta$; this implies $d(\delta d\zeta) = 0$; as $\delta d(d\zeta) = 0$, $d\zeta$ is therefore harmonic; as it is homologous to 0, it is 0. This shows that $d\zeta = 0$ and $\omega = d\delta\zeta$. As Δ commutes with C, we have $C\omega = \Delta(C\zeta)$; as $d(C\omega) = 0$, we can reason in the same manner on $C\omega$, $C\zeta$ as we have done on ω, ζ, and so we get $d(C\zeta) = 0$ and $C\omega = d\delta C\zeta$ (which proves again that $C\omega$ is homologous to 0). By another fundamental identity of the Kähler metric, we have $\Lambda d - d\Lambda = -C\delta C^{-1}$ (the nature of the operator Λ need not concern us here) and so $dC\delta = -d\Lambda dC$ and $dC\delta\zeta = -d\Lambda(dC\zeta) = 0$. By Stokes's theorem,

we have $\int_Q \omega = \int_X \delta\zeta$. But if λ is any form of degree $2p$, λ and $C\lambda$ induce the same form on every analytic subvariety of V of complex dimension p; in particular, we have $\int_X \delta\zeta = \int_X C\delta\zeta = \int_Q d(C\delta\zeta) = 0$. This shows that $\int_Q \omega$ depends only upon x'; as it is a linear function of x', it can be written as $\langle x', x\rangle$, where $x = x(Q)$ is an element of the homology group A_0 of V with real coefficients for the dimension $2p+1$. If X is given, Q is well determined modulo cycles of dimension $2p+1$ with integral coefficients; if Z is such a cycle, its class z in A_0 belongs to the subgroup of A_0 which we have called Δ, or more precisely $\Delta^{(p)}$; if we replace Q by $Q+Z$, $x(Q)$ is replaced by $x(Q+Z) = x(Q) + z$. Therefore, when X is given, $x(Q)$ is well determined modulo Δ; if we denote by $c_p(X)$ the image of $x(Q)$ in the factor-group A_0/Δ, $X \to c_p(X)$ is therefore a homomorphism of $\mathfrak{G}_h^{(p)}$ into Θ_p whose definition depends only upon the complex structure of V provided there exists at least one Kähler metric on V. This is the canonical mapping that we wished to construct.

For $p = 0$, the group $\mathfrak{G}_h^{(0)}$ is the group generated by the cycles $M - P$ if P is a fixed point on V and M is arbitrary; in order to determine c_0, it is therefore enough to know the mapping $M \to c_0(M - P)$ of V into Θ_0; it is easy to see that this is no other than the canonical mapping F which was defined in § I, no. 7. On the other hand, it follows from a theorem of Kodaira (Theorem 3, p. 110, in [3]) that the mapping c_{n-1} of $\mathfrak{G}_h^{(n-1)} = \mathfrak{G}_a$ into Θ_{n-1} is the same as the canonical mapping c defined above in § I, no. 12.

It is of course tempting to conjecture that any analytic family of cycles of dimension p on V, belonging to the group $\mathfrak{G}_h^{(p)}$, is mapped analytically into Θ_p by c_p; this seems very likely, but I have not yet succeeded in proving it. One would also like c_p to map $\mathfrak{G}_h^{(p)}$ onto Θ_p, as this would pave the way to an algebraic interpretation of the Jacobian varieties Θ_p when V is an algebraic variety; in fact, Θ_p could then be described as the factor-group of $\mathfrak{G}_h^{(p)}$ modulo a concept of equivalence which one might hope to define in algebraic terms. But Prof. Hodge has pointed out to me that c_p cannot map $\mathfrak{G}_h^{(p)}$ onto Θ_p unless all harmonic forms of degree $2p+1$ are sums of forms of type $(p+1, p)$ and of forms of type $(p, p+1)$; as in general this is not the case, some altogether new idea is required if the higher Jacobian varieties are to acquire the place which seems to belong to them in algebraic geometry.

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ON PICARD VARIETIES.*

By Wei-Liang Chow.

1. Introduction. In a recent paper [1] J. Igusa has developed a theory of the Picard variety attached to a non-singular algebraic variety V. Let $\mathfrak{G}_c(V)$ be the group of all divisors in V which are continuously equivalent to zero, and let $\mathfrak{G}_l(V)$ be the group of all divisors in V which are linearly equivalent to zero; let A be the Abelian variety determined by the periods of the Abelion integrals of the first kind on V, and let P be the dual Abelian variety of A in the sense of Weil (see section 3). The variety P is called the Picard variety attached to V, and Igusa's main result is that the group $\mathfrak{G}_c(V)$ can be mapped homomorphically onto the Picard variety P such that the kernel of this homomorphism is precisely the group $\mathfrak{G}_l(V)$. A. Weil [2] has taken up the same problem (for the possibly more general case of a Hodge variety) by a quite different method and has obtained among others also this result of Igusa, which he calls the first duality theorem of Igusa. There is however one important point concerning this theorem which is not settled in Igusa's paper; this is the question whether the mapping of $\mathfrak{G}_c(V)$ onto P is an analytic mapping in the sense (to be defined more precisely later in section 2) that it is analytic for every analytic system of divisors in $\mathfrak{G}_c(V)$. The importance of this question has been recognized by Weil, who considers the solution of this question, which he has obtained by a very elegant method, as the Main Theorem of his theory.[3] In this paper we shall give a new, simple proof of Igusa's duality theorem, which will at the same time take care of this additional question of analyticity almost without any further effort. Our method, like that of Igusa, is based on the existence of a "sufficiently general" curve in V, but we shall make systematic use of the concept of

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[1] J. Igusa, "On the Picard varieties attached to algebraic varieties," *American Journal of Mathematics*, vol. 74 (1952), pp. 1-22.

[2] A. Weil, "On Picard varieties," *American Journal of Mathematics*, vol. 74 (1952), pp. 865-894.

[3] A. Weil, *loc. cit.*, § III. Weil's proof of this theorem holds only for the case when the parameter variety is an analytic manifold (i. e. a variety without multiple points); however, it can be extended to the general case of an arbitrary analytic parameter variety by using some of the concepts and results developed in our (unpublished) paper cited in footnote 10.

duality between Abelian varieties introduced by Weil.[4] Our theory holds only for an algebraic variety and is hence possibly less general than that of Weil; as a compensation, we are able to formulate and prove the analyticity of the mapping in a somewhat stronger sense than that given by Weil's theory.[5] In the last section we shall deal with the more algebraic aspects of the problem; on the basis of some results on the Jacobian varieties proved by us elsewhere,[6] we investigate the fields of definition of some of the varieties and transformations introduced in the previous section. These results enable us to construct in a very simple way a system of divisors which represents the group $\mathfrak{G}_c(V)/\mathfrak{G}_l(V)$ in a "generally" one-to-one manner. This is however a weaker result than that obtained by Weil,[7] who has constructed a system of divisors which represents $\mathfrak{G}_c(V)/\mathfrak{G}_l(V)$ in an exactly one-to-one manner. Finally, we should like to mention that the method used in this last section has far greater potentialities than the immediate objective there indicates; in fact, this method is part of an idea which can be developed into a theory of Picard varieties for algebraic varieties over an arbitrary ground field, as will be shown in a forthcoming paper of ours on this subject.

We add here a few remarks concerning terminology. In this paper, the expression "analytic" shall always mean complex-analytic; and, unless explicitly stated to the contrary, the dimension is always the complex dimension, which is twice the topological dimension. By a cycle is always meant an algebraic cycle, except when explicitly stated to be a topological cycle, in which case the dimension is of course understood to be topological. When we are dealing with mappings of group varieties into one another, the expressions homomorphism, isomorphism, automorphism, etc. shall always mean analytic or algebraic homomorphism, isomorphism, automorphism, etc., according as we are dealing with analytic or algebraic varieties. The possibility of any confusion on this score is excluded by the theorem [8] that any (closed) analytic subvariety of an algebraic variety or analytic correspondence of an algebraic variety into another is an algebraic subvariety or an algebraic correspondence respectively.

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[4] A. Weil, *loc. cit.*, § I.

[5] We shall prove that the mapping is holomorphic in the sense defined in section 2, while Weil's method (extended as indicated in footnote 3) yields only the somewhat weaker result that the mapping is analytic.

[6] W. L. Chow, "The Jacobian variety of an algebraic curve," to appear in print soon.

[7] A. Weil, *loc. cit.*, § II.

[8] W. L. Chow, "On compact complex analytic varieties," *American Journal of Mathematics*, vol. 71 (1949), pp. 893-914, Theorem V and Theorem VII.

2. Analytic mapping of cycles. By means of the method of associated forms [9] the aggregate of all positive cycles of a given dimension d and a given degree m (i. e. the degree in the ambient projective space) in an algebraic variety V can be represented in a one-to-one and algebraic manner by the points of an (in general reducible) algebraic variety, which we shall denote by $M(V; d, m)$. In fact, the (homogeneous) coordinates of a point in $M(V; d, m)$ can be taken to be the coefficients of the associated form of the positive cycle corresponding to it. We shall call this representation the canonical mappings of the positive cycles into $M(V; d, m)$. The canonical mapping has the property that if an analytic system of positive cycles $|Z|$ of dimension d and degree m in V is parametrized by an analytic variety W, i. e. if there is a closed analytic correspondence [10] between W and V such that to each point in W corresponds (with a suitable definition of multiplicity) a positive cycle of dimension d and degree m, then this parametrization induces an everywhere single-valued analytic transformation of W into $M(V; d, m)$. In case this last transformation is a holomorphic transformation of W into V, then $|Z|$ is said to be regularly parametrized by W. It is easily seen that if W is a manifold (i. e. a variety without multiple points), then the parametrization is always regular. If $|Z_1|$ and $|Z_2|$ are two analytic systems of positive cycles of the same dimension d and of degrees m_1 and m_2 respectively, both parametrized by the same analytic variety W, then the difference system $|Z_1 - Z_2|$, consisting of all

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[9] W. L. Chow and B. L. van der Waerden, "Zur algebraischen Geometrie. IX. Ueber zugeordnete Formen und algebraische Systeme von algebraischen Mannigfaltigkeiten," *Mathematische Annalen*, Band 113 (1937), pp. 692-704; see also B. L. van der Waerden, *Einfuehrung in die algebraische Geometrie*, §§ 36-7, pp. 153-162. More precisely, it is proved there that there is an algebraic correspondence (which is in general reducible) between $M(V; d, m)$ and V such that to each point of $M(V; d, m)$ there corresponds under this correspondence a positive cycle of dimension d and degree m in V, the multiplicities of the components of the cycle being determined by the associated form corresponding to the point in $M(V; d, m)$, and that in this way all positive cycles of dimension d and degree m in V can be obtained. This is what is meant by the statement that the positive cycles of dimension d and degree m in V are represented by the points of $M(V; d, m)$ in a one-to-one and algebraic manner.

[10] We shall use in this section some of the concepts developed in our paper, "On complex analytic varieties," to appear soon in this Journal. We refer to that paper for the definition of a complex analytic variety in abstracto and the definition of a holomorphic function on such a variety. We mention here that a closed analytic correspondence of an analytic variety W into an analytic variety V is defined by a closed analytic subvariety in the product variety $W \times V$, and that such a correspondence is called an everywhere single-valued analytic transformation, if to each point in W corresponds exactly one point in V. A single-valued analytic transformation of W into V is said to be a holomorphic transformation, if every holomorphic function on V is carried by the inverse transformation into a holomorphic function on W.

cycles $Z_1 - Z_2$ as the common parameter runs through the variety W, is said to be an analytic system of cycles of dimension d and degree $m = m_1 - m_2$, and W is its parameter variety. The parametrization of $|Z_1 - Z_2|$ by W is said to be regular, if both $|Z_1|$ and $|Z_2|$ are regularly parametrized by W.

Suppose we have a mapping $f(Z)$, which assign to each element of an aggregate $\{Z\}$ of cycles of a given dimension d in V a point in an analytic variety U. We shall say that this mapping $f(Z)$ is analytic, if for any analytic system of cycles in $\{Z\}$ parametrized by an analytic variety W, the mapping $f(Z)$ induces an (everywhere single-valued) analytic transformation of W into U. The mapping $f(Z)$ is said to be holomorphic (or regularly analytic) if this induced transformation is holomorphic whenever W is a regular parametrization of the analytic system. Assume that U is an Abelian variety and that the mapping $f(Z)$ is a homomorphism of a subgroup $\{Z\}$ of the group of all cycles of dimension d in V into U; then the mapping $f(Z)$ is essentially determined by the mapping of the positive cycles in $\{Z\}$. It is then easily seen that the mapping $f(Z)$ is holomorphic, if for the positive cycles of degree m in $\{Z\}$ (for any m) the mapping $f(Z)$ is the product of the canonical mapping of Z into $M(V; d, m)$ and a holomorphic transformation of a subvariety of $M(V; d, m)$ into U.

Let Y be an algebraic subvariety of dimension $r - d$ in V (r being the dimension of V) which is the intersection of V with a linear subspace L of the proper dimension in the ambient space, and let $\{Z\}$ be the aggregate of all positive cycles of dimension d and degree m in V which has a proper intersection with Y. Let $f(Z)$ be the mapping of $\{Z\}$ in $M(Y; 0, m)$ which is the product of the mapping $Z \to Z \cdot Y$ and the canonical mapping of $Z \cdot Y$ into $M(Y; 0, m)$; then $f(Z)$ is a holomorphic mapping. In fact, the mapping $Z \to Z \cdot Y$ induces a mapping f' of an open subset [11] in $M(V; d, m)$ into

[11] This subset is not only open in the sense of the ordinary topology, but also in the sense of the topology where the closed subsets are the (reducible or irreducible) algebraic subvarieties. To show this, we recall the fact that if $F(u^{(0)}, u^{(1)}, \cdots, u^{(d)})$ is the associated form of a positive cycle Z of dimension d and degree m and if $v^{(i)} = (v^{(i)}_0, v^{(i)}_1, \cdots, v^{(i)}_n)$, $i = 0, 1, \cdots, d$, are the hyperplane coordinates of any $d+1$ hyperplanes in the ambient projective space, then we have $F(v^{(0)}, v^{(1)}, \cdots, v^{(d)}) = 0$ if and only if these $d+1$ hyperplanes have a common intersection point with Z. See, e.g., B. L. van der Waerden, *loc. cit.*, p. 157. Now, if we let $v^{(1)}, \cdots, v^{(d)}$ be the hyperplane coordinates of any d hyperplane whose intersection is the linear subspace L, then $F(u^{(0)}, v^{(1)}, \cdots, v^{(d)})$ will vanish identically in the variables $u^{(0)}$ if and only if the cycle Z does not intersect L and hence also Y properly. The condition that $F(u^{(0)}, v^{(1)}, \cdots, v^{(d)})$ vanishes identically in the variables $u^{(0)}$ can obviously be expressed as a system of homogeneous algebraic equations in the coefficients of the form $F(u^{(0)}, u^{(1)}, \cdots, u^{(d)})$, which are the coordinates of the point in $M(V; d, m)$ representing the

$M(Y; 0, m)$; and since $f(Z)$ is evidently the product of the canonical mapping of Z into $M(V; d, m)$ and f', it is sufficient to show that this mapping f' is a holomorphic transformation. Now, the coordinates of the point in $M(V; d, m)$ representing a cycle Z are the coefficients of the associated form of Z, and the associated form of $Z \cdot Y$ can be obtained by substituting for certain sets of variables in the associated form of Z the coefficients of the linear equations defining the linear subspace L. Furthermore, by a well known property of the associated form mentioned in footnote 11, the so obtained associated form of $Z \cdot Y$ cannot vanish identically so long as Z intersects Y properly. As the coefficients of the associated form of $Z \cdot Y$ are the coordinates of the point in $M(Y; 0, m)$ representing $Z \cdot Y$, it follows that the mapping f' is a holomorphic transformation.

3. Dual pairing of complex tori. Following Weil we shall introduce the concept of duality between complex tori of a given dimension; a dual pair of complex tori is characterized by the properties that the one is the character group of the fundamental group of the other and that the automorphism of the covering group of the one corresponding to the multiplication by $\sqrt{-1}$ induces the same automorphism of the covering group of the other. We can describe this concept in non-invariant terms as follows. Let S be the complex vector space of dimension p, and let a coordinate system be chosen as that a vector in S is represented by a point $(z) = (z_1, \cdots, z_p)$ in the p-dimensional complex number space. Let $\mathfrak{P}$ be a $(2p, p)$-matrix whose rows represent $2p$ real-linearly independent vectors, and let $[\mathfrak{P}]$ be the subgroup of S generated by these $2p$ vectors. Then the quotient group $U = S/[\mathfrak{P}]$ is a complex torus of dimension p. Let S' be another complex vector space of dimension p, and let a coordinate system be chosen so that a vector in S' is also represented by a point $(z') = (z'_1, \cdots, z'_p)$ in the p-dimensional complex number space. We shall consider S' to be invariantly associated with S in such a way that if S undergoes a coordinate transformation $(z) \to (z)\mathfrak{T}$, then S' undergoes the coordinate transformation $(z') \to (z')\,{}^t\bar{\mathfrak{T}}^{-1}$. Let $\mathfrak{P}'$ be the $(2p, p)$-matrix such that $(\mathfrak{P}'\mathfrak{P}')\,{}^t(\mathfrak{P}\bar{\mathfrak{P}}) = 1$; then the quotient group $U' = S'/[\mathfrak{P}']$ is a complex torus of dimension p which is invariantly associated with U and which can be considered as the character group of $[\mathfrak{P}]$ by assigning to each point in U', represented by a point (z') mod. $[\mathfrak{P}']$, the character

$$\tfrac{1}{2}(\bar{z}')\,{}^t(z) + \tfrac{1}{2}(z')\,{}^t(\bar{z}) \qquad \text{(mod. 1)} \tag{1}$$

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cycle Z; it follows then that the set of all points in $M(V; d, m)$ which represents cycles having improper intersections with Y is an (in general reducible) algebraic subvariety in $M(V; d, m)$.

for all points (z) in $[\mathfrak{P}]$. It is clear that this relation between U and U' is mutual; they are called the dual of each other, and we shall say that they are dually paired by the form (1).

The duality between U and U' can be extended to the (closed, connected) analytic subgroups and the corresponding factor groups, which are also all complex tori. Let U_0 be an analytic subgroup of dimension $p-q$ in U; then the component of the trace of U_0 in S which contains the zero vector is also an analytic subgroup and hence must be a linear subspace S_0 of dimension $p-q$, so that we have $U_0 = S_0/S_0 \cap [\mathfrak{P}]$ and hence U_0 is a complex torus. Without any loss of generality, we can assume the coordinate system of S so chosen that S_0 is the subspace defined by $z_1 = \cdots = z_q = 0$; let S_1 be the subspace defined by $z_{q+1} = \cdots = z_p = 0$. Similarly, let S'_0 be the subspace of S' defined by $z'_1 = \cdots = z'_q = 0$, and let S'_1 be the subspace defined by $z'_{q+1} = \cdots = z'_p = 0$. The fact that $S_0/S_0 \cap [\mathfrak{P}]$ is a compact manifold implies that the group $S_0 \cap [\mathfrak{P}]$ contains $2p-2q$ real-linearly independent vectors; hence, by a well known property of Abelian groups, it must possess a base which can be extended to a base of $[\mathfrak{P}]$. By a change of base of $[\mathfrak{P}]$ if necessary (which corresponds to the multiplication of $\mathfrak{P}$ by a unimodular integral matrix on the left), we can therefore assume that the last $2p-2q$ rows of $\mathfrak{P}$ constitute a base of $S_0 \cap [\mathfrak{P}]$. The matrix $\mathfrak{P}$ will then have the form $\begin{pmatrix} \mathfrak{P}_1 & * \\ 0 & \mathfrak{P}_0 \end{pmatrix}$, so that we have $S_0 \cap [\mathfrak{P}] = [\mathfrak{P}_0]$, and the corresponding matrix $\mathfrak{P}'$ will have the form $\begin{pmatrix} \mathfrak{P}'_1 & 0 \\ * & \mathfrak{P}'_0 \end{pmatrix}$. The relation $(\bar{\mathfrak{P}}'\mathfrak{P}')\,{}^t\mathfrak{P}\bar{\mathfrak{P}}) = 1$ implies then the relations $\bar{\mathfrak{P}}'_1\,{}^t\mathfrak{P}_1 + \mathfrak{P}'_1\,{}^t\bar{\mathfrak{P}}_1 = 1$ and $\bar{\mathfrak{P}}'_0\,{}^t\mathfrak{P}_0 + \mathfrak{P}'_0\,{}^t\bar{\mathfrak{P}}_0 = 1$; this shows that the rows of each of the matrices $\mathfrak{P}_1$, $\mathfrak{P}'_0$, $\mathfrak{P}'_1$ are also real-linearly independent and hence the groups $U_1 = S_1/[\mathfrak{P}_1]$, $U'_0 = S'_0/[\mathfrak{P}'_0]$, $U'_1 = S'_1/[\mathfrak{P}'_1]$ are all complex tori (of dimensions q, $q-p$, q respectively). Furthermore, these relations show that both U_0, U'_0 and U_1, U'_1 can be made into dual pairs, if we define the character relations in each pair by the same form (1) as restricted to the pairs of subspaces S_0, S'_0 and S_1, S'_1 respectively. The fact that the rows of $\mathfrak{P}'_0$ are real-linearly independent shows that $S'_1 \cap [\mathfrak{P}'] = [\mathfrak{P}'_1]$, and hence $U'_1 = S'_1/S'_1 \cap [\mathfrak{P}']$ can be considered as an analytic subgroup of U'. It is also clear that U_1 is isomorphic to the factor group U/U_0 and U'_0 is isomorphic to the factor group U'/U'_1, so that we can set $U_1 = U/U_0$ and $U'_0 = U'/U'_1$ and thus obtain the dual pairs U_0, U'/U'_1 and U/U_0, U'_1. Finally, we observe that U'_1, considered as a subgroup of the character group U' of $[\mathfrak{P}]$, is precisely the annihilator of $[\mathfrak{P}_0]$, and hence can be considered as the character group of $[\mathfrak{P}]/[\mathfrak{P}_0]$ under the canonical

homomorphism of $[\mathfrak{P}]$ onto $[\mathfrak{P}]/[\mathfrak{P}_0]$; and that if we identify $[\mathfrak{P}]/[\mathfrak{P}_0]$ with $[\mathfrak{P}_1]$ by projecting S onto the subspace S_1, then for each element in U'_1 represented by a point (z') mod. $[\mathfrak{P}'_1]$, this induced character is defined by the form (1) for all (z) in $[\mathfrak{P}_1]$. Thus the dual pairing of U/U_0 and U'_1 is an analytic extension of the well known group-theoretic pairing of $[\mathfrak{P}]/[\mathfrak{P}_0]$ and U'_1, and we shall express this fact by saying that the dual pairing of U/U_0 and U'_1 is subordinate to the given dual pairing of U and U'. Similarly, the dual pairing of U_0 and U'/U'_1 if of course also subordinate to the dual pairing of U and U'.

A complex torus U is said to be an Abelian variety, if U is an algebraic variety, i. e., more precisely, if there is a one-to-one bi-holomorphic transformation of U onto an algebraic variety in some projective space. It follows from this definition and the theorem [8] quoted in section 1 that any analytic subgroup U_0 in an Abelian variety U is also an Abelian variety. Furthermore, it is known from the theory of Riemann matrices that the dual of an Abelian variety is also an Abelian variety; it follows then from this and the above paragraph that also the factor group U/U_0 is an Abelian variety.[12]

4. The Jacobian varieties. We shall give in this section a brief resumé of some properties of the Jacobian varieties which will be useful to us later. Let C be a non-singular algebraic curve of genus p. Let $(\alpha) = (\alpha_1, \cdots, \alpha_{2p})$ be a base of its first Betti group $\mathfrak{B}(C)$, and let I_α be the intersection matrix of this base. Let $(\int\Phi) = (\int\Phi_1, \cdots, \int\Phi_p)$ be a system of p independent Abelian integrals of the first kind on C, and let Ω be the $(2p, p)$-matrix whose i-th row is the vector $(\int_{\alpha_i}\Phi_1, \cdots, \int_{\alpha_i}\Phi_p)$. The complex torus $J = S/[\Omega]$ is called the Jacobian variety of C, and the dual complex torus $J' = S'/[\Omega']$ is called the dual Jacobian variety of C; it is known that J is an Abelian variety and that J' is isomorphic to J and hence is also an Abelian variety. In fact, if we set $\mathfrak{R} = {}^t\bar{\Omega}I_\alpha^{-1}\Omega$, then we will have from the known properties

[12] It can be shown by the method of associated forms that the factor group U/U_0 of an Abelian variety U over an Abelian subvariety (or a finite subgroup) U_0 is also an Abelian variety. More precisely, we have the following theorem: Let T be a homomorphism of an Abelian variety U into an Abelian variety U', and let K be a field over which U, U' and T are all defined; then the kernel U_0 of T is a rational cycle over K and the factor group U/U_0 is an Abelian variety which can be represented by an algebraic variety defined over K; furthermore, the homomorphism T induces then an isomorphism of the so represented Abelian variety U/U_0 into U', which is also defined over K. The proof of this theorem will be given in our forthcoming paper mentioned in section 1.

of the Riemann matrix Ω the relation $I_a^{-1}\Omega = \Omega'\mathfrak{N}$; this shows that the linear transformation $(z) = (z')\mathfrak{N}$ of S' onto S induces an isomorphism of J' onto J, which we shall denote by N.

Let x_0 be any fixed point in C; then for any point x in C the mapping

$$f(x) = (\int_{x_0}^{x_1} \Phi_1, \cdots, \int_{x_0}^{x_1} \Phi_p) \qquad (\text{mod. } [\Omega])$$

is a holomorphic transformation of C into J, which can be extended to any 0-cycle $Z = \Sigma\, n_i x_i$ in C by the stipulation $f(Z) = \Sigma\, n_i f(x_i)$. This mapping $f(Z)$ is holomorphic. To show this we can without any loss of generality restrict ourselves to the case where Z is a positive 0-cycle of a given degree m in C. Then the corresponding variety $M(C; 0, m)$ is simply the m-th symmetric product of the curve C and hence is a non-singular algebraic variety.[13] The mapping $f(Z)$ induces an everywhere single-valued analytic transformation of $M(C; 0, m)$ into J, which on account of the non-singular nature of $M(C; 0, m)$ must be a holomorphic transformation. As $f(Z)$ is evidently the product of the canonical mapping of Z into $M(C; 0, m)$ and this last holomorphic transformation, our assertion is therefore proved.

Let $\int \Psi$ be the Abelian integral of the third kind on C which has the 0-cycle $x - x_0$ as its residue divisor and whose periods are all pure imaginary. We can then define a character of $\mathfrak{B}(C)$ by assigning to each element α in $\mathfrak{B}(C)$ the real number $\frac{1}{2\pi\sqrt{-1}} \int_\alpha \Psi (\text{mod. } 1)$. If we identify $\mathfrak{B}(C)$ with $[\Omega]$ by means of the isomorphic mapping $\sum_{i=1}^{2p} m_i\alpha_i \rightarrow (m)\Omega$, then this character is represented by a point $g(x)$ in J'. In fact, we have the formula

$$g(x) = \frac{1}{2\pi\sqrt{-1}} (\int_{a_1} \Psi, \cdots, \int_{a_{2p}} \Psi)\Omega' \qquad (\text{mod. } [\Omega']),$$

and this mapping $g(x)$ of C into J' can be extended to any 0-cycle $Z = \Sigma\, n_i x_i$ in C by the stipulation $g(Z) = \Sigma\, n_i g(x_i)$. It is clear that $g(Z) = 0$ (mod. $[\Omega']$) if and only if Z is a principal divisor, so that $g(Z)$ induces an isomorphic mapping of $\mathfrak{G}_c(C)/\mathfrak{G}_l(C)$ into J'. Furthermore, this mapping $g(Z)$ is also holomorphic. In fact, let $\int \phi_1, \cdots, \int \phi_{2p}$ be the harmonic integrals on C corresponding to the topological 1-cycles $\alpha_1, \cdots, \alpha_{2p}$ respectively, and let $\mathfrak{M}$ be a $(p, 2p)$-matrix such that

$$(\int \phi_1, \cdots, \int \phi_{2p}) = (\int \bar{\Phi}_1, \cdots, \int \bar{\Phi}_p, \int \Phi_1, \cdots, \int \Phi_p) \begin{pmatrix} \bar{\mathfrak{M}} \\ \mathfrak{M} \end{pmatrix}.$$

[13] See B. L. van der Waerden, "Divisorklassen in algebraischen Funktionenkoerpern," *Commentarii Mathematici Helvetici*, vol. 20 (1947), pp. 68-109, § 13.

Then we have

$$g(x) = \left(\int_{x_0}^{x} \phi_1, \cdots, \int_{x_0}^{x} \phi_{2p}\right)\Omega'$$

$$= \left(\int_{x_0}^{x} \bar{\Phi}_1, \cdots, \int_{x_0}^{x} \bar{\Phi}_p, \int_{x_0}^{x} \Phi_1, \cdots, \int_{x_0}^{x} \Phi_p\right)\begin{pmatrix}\mathfrak{M}\\ \bar{\mathfrak{M}}\end{pmatrix}\Omega'$$

$$= \left(\int_{x_0}^{x} \bar{\Phi}_1, \cdots, \int_{x_0}^{x} \bar{\Phi}_p, \int_{x_0}^{x} \Phi_1, \cdots, \int_{x_0}^{x} \Phi_p\right){}^t(\Omega'\bar{\Omega}')I_a\Omega'$$

$$= \left(\int_{x_0}^{x} \Phi_1, \cdots, \int_{x_0}^{x} \Phi_p\right)\mathfrak{N}^{-1} \qquad (\text{mod. } [\Omega']);$$

this shows that $g(x) = N^{-1}f(x)$, which proves our assertion. Thus in the isomorphism N of J' onto J, the mapping $g(x)$ is carried over into the mapping $f(x)$; this shows that these two mappings are essentially the same, being given here two different interpretations as mappings into a pair of dual Abelian varieties.

5. The Picard varieties. Let V be a non-singular algebraic variety of dimension r embedded in a projective space, and let C be a non-singular algebraic curve in V such that any topological 1-cycle in V is homologous to a topological 1-cycle in C. Such a curve C certainly exists; we can take, for example, the intersection of V with a sufficiently general linear subspace of the complementary dimension in the ambient space of V.[14] If $2q$ is the rank of the first Betti group $\mathfrak{B}(V)$ of V, then there exist (according to a theorem of Hodge) exactly q independent Abelian integrals of the first kind $\int\Theta_1, \cdots, \int\Theta_q$ in V. Let $(\gamma) = (\gamma_1, \cdots, \gamma_{2q})$ be a base of $B(V)$, and let Ω_1 be the $(2q, q)$-matrix whose i-th row is the vector $\left(\int_{\gamma_i}\Theta_1, \cdots, \int_{\gamma_i}\Theta_q\right)$. Since the rows of Ω_1 are real-linearly independent, the quotient group $S_1/[\Omega_1]$ is a complex torus of dimension q, which we shall call the Albanese variety A attached to V. The dual complex torus P of A is then called the Picard variety attached to V. Let x_0 be any fixed point in V, which we shall assume for the sake of convenience to be also a point in C and take as the reference

[14] That such an intersection has the indicated property is usually given a very sketchy proof in the literature. For a more detailed treatment, see W. L. Chow, "On the fundamental group of an algebraic variety," *American Journal of Mathematics*, vol. 74 (1952), pp. 726-736, Theorem 1.

point in the definition of the mappings $f(x)$ and $g(x)$ in the previous section; then, for any point x in V the mapping

$$F(x) = \left(\int_{x_0}^{x} \Theta_1, \cdots, \int_{x_0}^{x} \Theta_q\right) \qquad (\text{mod. } [\Omega_1])$$

is a holomorphic transformation of V into A, which can be extended to any 0-cycle $Z = \Sigma\, n_i x_i$ in V by the stipulation $F(Z) = \Sigma\, n_i F(x_i)$. It is easily seen that $F(Z)$ is an analytic mapping of the 0-cycles in V into A; however, in contrast to the case of a curve, we do not know whether this mapping is also holomorphic.

The inclusion mapping of C into V induces a homomorphism L of $\mathfrak{B}(C)$ into $\mathfrak{B}(V)$, which can be expressed by the relation $(\alpha) \approx (\gamma)\mathfrak{L}$ on V, where $\mathfrak{L}$ is an integral $(2q, 2p)$-matrix. The assumption that every topological 1-cycle in V is homologous to a topological 1-cycle in C implies that this homomorphism L is onto, i. e. there exixsts an integral $(2p, 2q)$-matrix $\mathfrak{L}'$ such that $\mathfrak{L}\mathfrak{L}' = 1$. Since the integrals $\int\Theta_1, \cdots, \int\Theta_q$ are also Abelian integrals of the first kind on the curve C, there exists a (p, q)-matrix $\mathfrak{T}$ such that the relation $(\int\Theta) = (\int\Phi)\mathfrak{T}$ holds on the curve C. Integrating both sides over the base (α), we obtain the equation ${}^t\mathfrak{L}\Omega_1 = \Omega\mathfrak{T}$ and hence the equation $\Omega_1 = {}^t\mathfrak{L}'\Omega\mathfrak{T}$; we have then the equation $[\Omega\mathfrak{T}] = [\Omega_1]$ and hence the mapping $(z) \to (z)\mathfrak{T}$ induces a homomorphism T of J onto A. The kernel J_0 of T is an analytic subgroup of dimension $p - q$ in J; furthermore, if we denote by S_0 the linear subspace of dimension $p - q$ in S defined by the equation $(z)\mathfrak{T} = 0$, then we have evidently $J_0 = S_0/S_0 \cap [\Omega]$ and hence J_0 is an Abelian variety. The homomorphism T induces an isomorphism of the factor group J/J_0 onto A, which we shall also denote by T; since J/J_0 is an Abelian variety, it follows that A and consequently also P are Abelian varieties. Let the subgroup J'_1 in J' be the annihilator of $S_0 \cap [\Omega]$; we have shown before that J'_1 is the dual of J/J_0 and that the dual pairing of J/J_0 and J'_1 is subordinate to the dual pairing of J and J'. The isomorphism T of J/J_0 onto A then induces an isomorphism T' of P onto J'_1, such that the dual pairing of A and P is carried by these isomorphisms into the dual pairing of J/J_0 and J'_1.

Let D be a divisor in V which is homologous to zero. According to the Weil's generalization of a theorem of Lefschetz,[15] there exists an Abelian integral $\int\Psi$ of the third kind on V with D as its residue divisor. Adding a linear combination of the Abelian integrals of the first kind if necessary,

[15] A. Weil, "Sur la théorie des formes différentielles attachées à une variété analytique complexe," *Commentarii Mathematici Helvetici*, vol. 20 (1947), pp. 110-116.

we can assume that the periods of $\int\Psi$ are all pure imaginary; then the integral $\int\Psi$ is uniquely determined by the divisor D up to an additive constant. We can then define a character of $\mathfrak{B}(V)$ by assigning to each element γ in $\mathfrak{B}(V)$ the real number $\frac{1}{2\pi\sqrt{-1}}\int_\gamma \Psi \pmod 1$. If we identify $\mathfrak{B}(V)$ with $[\Omega_1]$ by means of the isomorphic mapping $\sum_{i=1}^{2q} m_i\gamma_i \rightarrow (m)\Omega_1$, then this character is represented by a point $G(D)$ in P. Thus we have a homomorphism $G(D)$ of the group $\mathfrak{G}_h(V)$ of all divisors in V which are homologous to zero, into the Abelian variety P. By considering the multiplicative function $\exp(\int_{x_0}^{x}\Psi)$ on V, it can be easily seen that the kernel of this homomorphism is precisely the group $\mathfrak{G}_l(V)$, so that $G(D)$ induces an isomorphism of $\mathfrak{G}_h(V)/\mathfrak{G}_l(V)$ into P. We shall show that this mapping $G(D)$ is holomorphic. We observe first that if the intersection 0-cycle $C\cdot D$ is defined, then the integral $\int\Psi$ can also be considered as an integral of the third kind on the curve C with $C\cdot D$ as its residue; and as such it defines a character of $\mathfrak{B}(C)$ which is the same as that induced by the corresponding character of $\mathfrak{B}(V)$ under the homomorphism L of $\mathfrak{B}(C)$ onto $\mathfrak{B}(V)$. We have then the equation $G(D) = T'^{-1}g(C\cdot D)$; we observe here that the mapping $T'^{-1}g(C\cdot D)$ is defined for any divisor D (not necessarily homologous to zero) such that $C\cdot D$ is defined. Now, let $D(w) = D_1(w) - D_2(w)$ be an analytic system of divisors in V, (regularly) parametrized by an analytic variety W, where $D_1(w)$ and $D_2(w)$ are positive divisors such that $D_1(w) \sim D_2(w)$ for all w. If w_0 is any point in W, we can choose the curve C such that the intersection 0-cycle $C\cdot D(w_0)$ is defined; then the intersection 0-cycle $C\cdot D(w)$ is defined for all points w in a sufficiently small neighborhood of w_0. Furthermore, we can assume that C is the intersection of V with a suitably chosen linear subspace in the ambient space. Then we have the equation

$$G(D(w)) = T'^{-1}g(C\cdot D(w)) = T'^{-1}g(C\cdot D_1(w)) - T'^{-1}g(C\cdot D_2(w))$$

for all points w in a neighborhood of w_0. The mapping $T'^{-1}g(C\cdot D_1(w))$ is obviously the product of the canonical mapping of $D_1(w)$ into $M(V; r-1, m)$ (m being the degree of $D_1(w)$), the holomorphic transformation of $M(V; r-1, m)$ into $M(C; 0, m)$ induced by the mapping $D\rightarrow C\cdot D$ (see the last paragraph of section 1), the holomorphic transformation of $M(C; 0, m)$ into J' induced by $g(Z)$, and the isomorphism T'^{-1} of J'_1 onto P; it follows then that $T'^{-1}g(C\cdot D_1(w))$ is an analytic (holomorphic) transformation of a neighborhood of w_0 into P, and similarly also $T'^{-1}g(C\cdot D_2(w))$

is an analytic (holomorphic) transformation. Thus we have shown that $G(D(w))$ is an analytic (holomorphic) transformation of a neighborhood of w_0 into P; and since w_0 is any point in W, it follows that $G(D(w))$ is an analytic (holomorphic) transformation of W into P. This proves our assertion.

In order to complete the proof of Igusa's duality theorem, it only remains to show that $\mathfrak{G}_h(V) = \mathfrak{G}_c(V)$ and that the mapping $G(D)$ of $\mathfrak{G}_h(V)$ into P is onto. Since $\mathfrak{G}_c(V)$ is a subgroup of $\mathfrak{G}_h(V)$, both assertions will follow if we show that $G(D)$ maps the group $\mathfrak{G}_c(V)$ onto P; and this can be done by the construction of the so-called Poincaré family of divisors. However, we shall construct in the next section an analytic system of divisors which represents the group $\mathfrak{G}_c(V)/\mathfrak{G}_l(V)$ in a more precise manner.

6. Representative system of divisors. We begin by observing (what we have implicitly assumed up to now) that the field $\mathfrak{R}$ of all complex numbers is the universal domain, in the sense of Weil,[16] of the algebraic geometry we are concerned with here. We shall assume that the Abelian variety A is represented by an arbitrarily chosen, but fixed non-singular algebraic variety in some projective space; it follows then that the mapping $F(x)$ of V into A is a rational transformation, defined (in the sense of Weil) for all points of V. Let K be a field (i. e., in the terminology of Weil, a subfield of $\mathfrak{R}$ such that $\mathfrak{R}$ has infinite degree of transcendency over K) over which the algebraic varieties V and A, as well as the rational transformation $F(x)$ and the group composition in A, are all defined; furthermore, we shall assume that the point x_0, the reference point in the definition of $F(x)$, is rational over K. We can, in fact, take K to be a finitely generated extension of the rational field. Consider an $(r-1)$-dimensional linear system $|C|$ of curves on V, which is cut out on V by a system of linear subspaces in the ambient space; by choosing the system of linear subspaces in a suitable manner we can assume that the system $|C|$ has the following properties: (1) the system $|C|$ is defined over K, i. e. the associated variety W of $|C|$ in $M(V; 1, m)$, m being the degree of $|C|$, is defined over K; for any point w in W, we shall denote by $C(w)$ the curve of the system corresponding to it; (2) the point x_0 is a base point of $|C|$, and a generic point of V over K is contained in exactly one curve of the system $|C|$; (3) for any generic point u of W over K, the curve $C(u)$ is a non-singular curve, whose genus

[16] A. Weil, *Foundations of Algebraic Geometry*, American Mathematical Society Colloguium Publication no. 29 (1946). We shall use in this section the algebraic-geometric concepts and terminology developed in this book.

will be denoted by p. According to the theorem cited in footnote 14, the curve $C(u)$ will have the property (stated at the beginning of section 5) that every topological 1-cycle in V is homologous to a topological 1-cycle in $C(u)$. We can therefore apply the results of sections 4 and 5 to the curve $C(u)$; let $J(u)$ be the Jacobian variety of $C(u)$ and let T_u be the homomorphism of $J(u)$ onto A defined in section 5. Since $C(u)$ is defined over $K(u)$, it follows from a result proved elsewhere [17] that we can consider $J(u)$ as represented by a non-singular algebraic variety, such that both the variety $J(u)$ as well as the group composition in $J(u)$ are defined over $K(u)$. Moreover, since the point x_0 is a rational divisor of degree 1 in $C(u)$ over $K(u)$, the "canonical mapping" f_u of $C(u)$ into $J(u)$, defined with x_0 as the reference point, is also a rational transformation defined over $K(u)$. Then the homomorphism T_u is a rational transformation of $J(u)$ onto A; we maintain that it is also defined over $K(u)$. In fact, the rational transformation $F(x)$ of V into A induces a rational transformation of $C(u)$ into A, which is evidently defined over $K(u)$. Since A is an Abelian variety, this rational transformation has a uniquely determined linear extension,[18] which is a rational transformation of $J(u)$ into A, also defined over $K(u)$; and it can be easily seen, in view of our choice of the reference point x_0 for both $F(x)$ and f_u, that this rational transformation is precisely the homomorphism T_u.

The fact that T_u is defined over $K(u)$ is the basis for the construction of what is usually called the Poincaré family of divisors,[19] which is "generically" parametrized by A. Though this family does contain a representative for each element in $\mathfrak{G}_c(V)/\mathfrak{G}_l(V)$, it has the disadvantage that this representative is in general not uniquely determined, even for a "generic" element in $\mathfrak{G}_c(V)/\mathfrak{G}_l(V)$, there being a finite number of them in the family; this is due to the fact that A can be represented as a factor group $J(u)/J_0(u)$ of $J(u)$, but not necessarily as a subgroup of $J(u)$. In order to get a "generally" one-to-one representative system of divisors for $\mathfrak{G}_c(V)/\mathfrak{G}_l(V)$, we shall show that the isomorphism T'_u of P into $J'(u)$, as defined in section 5, is also defined over $K(u)$, provided we represent both P and $J'(u)$ by suitably chosen algebraic varieties. In fact, since $J(u)$ and $J'(u)$ are

[17] See the paper cited in footnote 6.

[18] See A. Weil, *Variétés abéliennes et courbes algébriques*, p. 77, Théorème 21.

[19] We have in mind here the method of construction due to F. Enriques, assuming the existence of q independent Abelian integrals of the first kind in V. See O. Zariski, *Algebraic Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 3, no. 5 (1935), p. 123 and the references cited there.

isomorphic, we can represent both of them by the same algebraic variety and thus identify $J'(u)$ with $J(u)$ by means of the isomorphism N defined in section 4; we can then consider $J'_1(u)$ as an Abelian subvariety of $J(u)$ and we have evidently $g_u = f_u$. The construction of the algebraic variety representing P will be given in course of the proof below; we shall see that the so represented variety P is defined over K.[20]

Let X be a cycle of dimension q in $J(u)$, rational over $K(u)$; then for a generic point v of A over $K(u)$, the intersection $X \cdot T_u^{-1}(v)$ is a rational 0-cycle over $K(u, v)$. If for any 0-cycle Z in an Abelian variety we denote (with Weil [21]) by $\mathfrak{S}(Z)$ the element in the Abelian variety which is the sum of all points in Z, each point being added or subtracted as often as its (positive or negative) multiplicity indicates, then the point $\mathfrak{S}(X \cdot T_u^{-1}(v))$ in $J(u)$ is rational over $K(u, v)$; hence, the mapping $v \to \mathfrak{S}(X \cdot T_u^{-1}(v))$ is a rational transformation of A into $J(u)$, defined over $K(u)$. The mapping $H_u : v \to \mathfrak{S}(X \cdot T_u^{-1}(v)) - \mathfrak{S}(X \cdot T_u^{-1}(0))$, is then a homomorphism of A into $J(u)$, defined over $K(u)$, and hence the image $H_u(A)$ is an Abelian subvariety in $J(u)$, also defined over $K(u)$. It is clear that if s is the degree of the 0-cycle $X \cdot T_u^{-1}(v)$, then we have the equation $T_u H_u(v) = \mathfrak{S}(sv)$; this shows that $H_u(A)$ has the dimension q, and that the kernel A_0 of H_u is a finite subgroup of A and hence is a positive 0-cycle, rational over $K(u)$. Since the Abelian variety A is defined over K, the positive 0-cycle A_0, being a finite subgroup in A, must be algebraic over K; and as $K(u)$ is a regular extension of K, it follows that A_0 is rational over $K(u) \cap \bar{K} = K$. According to the theorem cited in footnote 12, the factor group A/A_0 is then an Abelian variety which can be represented by a non-singular algebraic variety, defined over K, and the homomorphism H_u of A into $J(u)$ induces an isomorphism T'_u of A/A_0 into $J(u)$, which is also defined over $K(u)$. We shall now show that $T'_u(A/A_0) = J'_1(u)$, so that A/A_0 is isomorphic to P and hence can be taken as a representative algebraic variety of P. In order to prove this, we only need to show that for a generic point y of A/A_0 over $K(u)$, there exists a divisor $D(y)$ in $G_c(V)$ such that $f_u(D(y) \cdot C(u)) = T'_u(y)$. We recall that $M(C; 0, p)$ is a non-singular (irreducible) algebraic variety of dimension p, defined over K; in fact, it is the symmetric p-th product of the curve $C(u)$.

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[20] The first part of the proof below, namely the construction of the Abelian variety $H_u(A)$, is essentially the same as Castelnuovo's proof for Poincaré's "complete reducibility theorem." See G. Castelnuovo, "Sugli integrali semplici appartenenti ad una superficie irregolare," *Rend. della R. Accad. dei Lincei*, ser. V, vol. XIV (1905); compare also A. Weil, *loc. cit.*, p. 94, Théorème 26.

[21] A. Weil, *loc. cit.*, p. 28.

The mapping f_u of the cycles in $C(u)$ into $J(u)$ induces a birational transformation h_u of $M(C;0,p)$ onto $J(u)$, defined over $K(u)$; hence, since $T'_u(y)$ is a generic point of $T'_u(A/A_0)$ over $K(u)$, the inverse image $h_u^{-1}(T'_u(y))$ is a rational point of $M(C;0,p)$ over $K(u,y)$ and represents a positive 0-cycle $Z(u,y)$ of degree p in $C(u)$, which is rational over $K(u,y)$. Over the field $K(y)$, the mapping $u \to Z(u,y)$ defines an algebraic correspondence of W into V, and it is easily seen that the projection of this correspondence in V is a positive divisor $Z(y)$ in V, rational over $K(y)$, whose intersection with $C(u)$ is precisely $Z(u,y)$, apart from a possible fixed component Z_0. Let $Z(0)$ be a specialization of $Z(y)$ over the specialization $y \to 0$ over K, such that $C(u) \cdot Z(0) = px_0 + Z_0$; if K' is an extension of K over which the divisor $Z(0)$ is rational, then $D(y) = Z(y) - Z(0)$ is a divisor in $\mathfrak{G}_c(V)$ such that

$$\begin{aligned} f_u(D(y)\cdot C(u)) &= f_u(Z(y)\cdot C(u)) - f_u(Z(0)\cdot C(u)) \\ &= f_u(Z(u,y)) - f_u(px_0) \\ &= f_u(h_u^{-1}(T'_u(y))) - f_u(px_0) = T'_u(y). \end{aligned}$$

This proves our assertion.

If we now identify P with A/A_0, then the divisor $D(y)$ will generate an analytic system of divisors in $\mathfrak{G}_c(V)$, "generically" parametrized by P in the sense that P is the parameter variety of this system for all generic points of P over K'; furthermore, if $D(y_0)$ is any specialization of $D(y)$ over any specialization $y \to y_0$ over K', then we have $G(D(y_0)) = y_0$. However, this system gives a one-to-one representation of the group $\mathfrak{G}_c(V)/\mathfrak{G}_1(V)$ only "in general," since it is possible that for some special point y_0 in P there exist more than one specializations of $D(y)$ over the specialization $y \to y_0$ over K'.

Finally, going over once more the arguments in this section, it will be readily seen that we have in fact proved the following result: If H is a homomorphism of any Abelian variety B into $J(u)$ and if B is defined over K and H is defined over $K(u)$, then $H(B)$ is contained in $J'_1(u)$. Thus we can characterize the Picard variety P as the "largest Abelian subvariety" in $J(u)$ which is defined over K. As mentioned in section 1, this result is part of a purely algebraic theory of Picard varieties for algebraic varieties over an arbitrary ground field, which will be developed in a forthcoming paper.

THE JOHNS HOPKINS UNIVERSITY.

ON k-FOLD IRREDUCIBILITY OF MAPPINGS.*

By G. T. WHYBURN.

1. Introduction. In an earlier paper [1] it was shown by the author that a mapping $f(A) = B$ on a compact set A is (strongly) irreducible if and only if the points x in A satisfying $x = f^{-1}f(x)$ are dense in A. In this paper a similar characterization of the property of being irreducibly k-fold will be established. As in the earlier theorem, the proof can be made to hinge on the upper semi-continuity of a strategically defined diameter function.

A mapping $f(A) = B$ is said to be *irreducibly k-fold, k* a positive integer, on a closed set X in A provided that for each $y \in B$, $X \cdot f^{-1}(y)$ contains at least k points but if X' is any closed proper subset of X, then $X' \cdot f^{-1}(y)$ contains less than k points for at least one $y \in B$. For $k = 1$ this is identical with ordinary irreducibility in the strong sense as referred to above. In what follows we shall assume that all spaces used are separable and metric. For any set Z, and positive number r, $V_r(Z)$ will denote the r-neighborhood of Z, i. e., the set of all points of the space at a distance $< r$ from the set Z.

2. The function $e_k(x)$. For any set A in a metric space and any positive integer k, let us define $e_k(A) = \text{g. l. b. } [\max \delta(A_i)]$ for all decompositions $A = \sum_1^k A_i$ of A into k non-empty subsets A_i. It will be noted at once that a set A is of power $\leqq k$ if and only if $e_k(A) = 0$. Also we remark that if k were allowed to assume the values ω (arbitrary finite number) and $\aleph_0$ we would have $e_\omega(A) = 0$ for all totally bounded sets A and, in a separable space, $e_{\aleph_0}(A) = 0$ for all sets A.

A mapping $f(X) = Y$ is said to generate an upper semi-continuous decomposition of X provided the decomposition of X into the sets $[f^{-1}(y)]$, $y \in Y$ is upper semi-continuous [2], i. e., the union of all sets of this collection contained in any open set in X is an open set in X.

THEOREM. *If the mapping $f(X) = Y$ generates an upper semi-continuous decomposition of X, the function $e_k(x) \equiv e_k[f^{-1}f(x)]$, $x \in X$, is upper semi-continuous.*

* Received April 21, 1952.

Proof. Let N be any real number and let U be the set of all $x \in X$ for which $e_k(x) < N$. We have to show that N is open. To that end let $x \in U$. If $3d = N - e_k(x)$ there exists a decomposition

$$A = f^{-1}f(x) = \sum_1^k A_i,$$

with $\delta(A_i) < N - 2d$ for $i = 1, \cdots, k$. This gives $V_d(A) = \sum_1^k V_d(A_i)$ with $\delta[V_d(A_i)] < \delta(A_i) + 2d < N$, for $i = 1, \cdots, k$. Now by hypothesis the union V of all sets $f^{-1}(y)$, $y \in Y$, contained in $V_d(A)$ is open. Further, any such set admits the decomposition

$$f^{-1}(y) = f^{-1}(y) \cdot V_d(A) = \sum_1^k f^{-1}(y) \cdot V_d(A_i)$$

with $\delta[f^{-1}(y) \cdot V_d(A_i)] < N$, $i = 1, 2, \cdots, k$, so that $e_k(v) < N$ for all $v \in V$. Accordingly U is open because $x \in V \subset U$.

3. Characterization of k-fold irreducibility. A mapping $f(X) = Y$ is at least k-fold provided $f^{-1}(y)$ contains at least k points for each $y \in Y$.

THEOREM. *If X is compact, an at least k-fold mapping $f(X) = Y$ is irreducibly k-fold on X if and only if the set of all $x \in X$ such that $f^{-1}f(x)$ consists of exactly k-points is dense in X.*

Proof. If the latter condition is satisfied, then for any closed proper subset X' of X, $X - X'$ would contain a point x for which $f^{-1}f(x)$ consists of just k points and thus so that $f^{-1}(x)$ has at most $k - 1$ points in X'. Hence f is irreducibly k-fold on X.

On the other hand suppose f is irreducibly k-fold on X. Let ϵ be any positive number and let U be any open set in X of diameter $< \epsilon$. Then since f fails to be at least k-fold on $X - U$, there exists a $y \in Y$ such that $X - U$ contains at most $k - 1$ points of $f^{-1}(y)$ and thus such that $f^{-1}(y)$ is contained in U plus the union of $k - 1$ additional points of X. Thus $f^{-1}(y)$ admits a decomposition into k sets each of diameter $< \epsilon$ so that $e_k(x) < \epsilon$ for each $x \in f^{-1}(y)$. Thus any open set U contains a point for which $e_k(x) < \epsilon$. By the upper semi-continuity of $e_k(x)$ established in § 2 and the compactness of X, it follows that the set of zeros of $e_k(x)$ is dense in X; and it has already been noted that the zeros of $e_k(x)$ are precisely the points x for which $f^{-1}f(x)$ consists of exactly k points.

4. Existence theorem. In general, for a mapping the property of being at least k-fold on a set is not inducible for $k > 1$. Hence the standard procedure, based on the Brouwer Reduction Theorem, for showing the existence of a set on which a property is present irreducibly is not available for $k > 1$ although it yields the expected result in case $k = 1$. However for finite-to-one mappings this procedure does apply as will now be shown.

LEMMA. *For finite-to-one mappings the property of being at least k-fold on a set is inducible.*

For let $f(X) = Y$ be a finite-to-one mapping and let $A_1 \supset A_2 \supset A_3 \supset \cdots$ be a monotone decreasing sequence of non-empty compact sets in X on each of which f is at least k-fold and let A be their intersection. If for some $y \in Y$, $A \cdot f^{-1}(y)$ contained $< k$ points, then since $f^{-1}(y)$ contains only a finite number of points and any point of $f^{-1}(y)$ not in A would fail to be in A_n for some n, then for n sufficiently large $A_n \cdot f^{-1}(y)$ would contain less than k points. As this contradicts our supposition, the lemma is proven.

The Brouwer Reduction Theorem now yields at once the following

THEOREM. *If X is compact and $f(X) = Y$ is any at least k-fold finite-to-one mapping, there exists a closed subset A of X on which f is an irreducibly k-fold mapping onto Y.*

For mappings which are not finite to one the existence question requires further study and it is proposed to deal with this in a later paper.

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REPRESENTATIONS OF PRIMES BY QUADRATIC FORMS.*

By N. C. ANKENY.[1]

Hecke proved that there exist infinitely many rational primes q such that

$$q = a^2 + b^2 \tag{1}$$

and $b^2 = o(q)$, as $q \to \infty$ (cf. [3]). In this paper we shall prove on the assumption of the Extended Riemann Hypothesis (E. R. H.) that there exist infinitely many rational primes q such that

$$q = a^2 + b^2 \tag{2}$$

and $b = O(\log q)$.

Hecke's method was the following: let α be any integer in the Gaussian Field F ($F = R(i)$, where R consists of the rational numbers). Now define the character $\chi(\alpha, n) = (\alpha/\bar{\alpha})^{2n}$, where n is any rational integer. Define

$$L(s, n) = \sum_{(\alpha)} \chi(\alpha, n)/N\alpha^s, \qquad \mathrm{Re}(s) > 1,$$

where (α) runs over all integral ideals in F. These new functions $L(s, n)$ define a new type of L-series which have many properties quite similar to the Dirichlet L-series. Hecke proved that the L-series had an Euler product and a functional equation.

Let $(P/\bar{P}) = e^{i\theta_P}$, where P runs over all prime numbers in F. Then, with the ordering of the prime ideals by the norms, $N_{FR}(P_i) \leqq {}_{FR}(P_{i+1})$

$$\theta_{P_1}, \theta_{P_2}, \theta_{P_3}, \cdots \tag{3}$$

are equidistributed $(\mathrm{mod}\ 2\pi)$. From this fact (1) immediately follows.

In the following we shall go further and prove a refinement of Hecke's results on the assumption of the E. R. H. We shall make use of the work of A. Selberg on the zeros of Dirichlet L-series (cf. [6], [7]).

We shall state here the two main theorems with regard to the equidistribution of (3).

Let $|\bar{u}|$ be defined as the absolute value of $\bar{u}$, where $u \equiv \bar{u} (\mathrm{mod}\ 2\pi)$,

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$-\pi \leqq \bar{u} \leqq \pi$. Define $f_1 = f_1(\theta, T, \lambda) = 1$ if $|\overline{\lambda - \theta}| \leqq (\log T)^{1+\epsilon} T^{-\frac{1}{2}}$; $f_1 = 0$ otherwise, where $\epsilon > 0$ and $T > T_0(\epsilon)$.

Let $N(P)$ be the norm from F to R of the ideal P.

THEOREM I. *If P runs over all prime ideals in F, then, for any λ,*

$$\sum_P f_1(\theta_P, T, \lambda)(\log N(P))e^{-N(P)/T} = 1/\pi T^{\frac{1}{2}}(\log T)^{1+\epsilon} + o(T^{\frac{1}{2}}(\log T)^{1+\epsilon}), \text{ as } T \to \infty.$$

Define $f_2 = f_2(\theta, T) = 1$ if $|\theta| \leqq c_1(\log T/T^{\frac{1}{2}})$; $f_2 = 0$ otherwise, where c_1 is a sufficiently large absolute constant. (This will be determined in Section **3**.)

THEOREM II. *If P runs over all prime ideals in F, then*

$$\sum_P f_2(\theta_P, T)(\log N(P))e^{-N(P)/T} > T^{\frac{1}{2}} \log T, \qquad T \to \infty.$$

The result (2) is derived in Section **3** from Theorem II.

Theorem I and II can be generalized to any quadratic extension of the rationals and this will be done in Section **4**. The result (2) can also be generalized (see Theorem III).

I would like to thank A. Selberg for suggestions on the presentation of this manuscript as well as actual sharpening of certain of the results.

Sections **1** and **2** contain prelimnary lemmas. Section **3** deals with the proofs of Theorems I and II. Section **4** gives an outline of the generalization of these results to other quadratic extensions of the rationals.

1. Define $g(x, \delta_1, \delta_2) = g(x)$ as follows:

(a) $g(x + 2\pi) = g(x)$, (b) $g(-x) = g(x)$,

(c) $g(x) = 1, 1 + 1/\delta_1(\delta_2 - x)$

for $0 \leqq x \leqq \delta_2$, $\delta_2 \leqq x \leqq \delta_1 + \delta_2$, $\delta_2 + \delta_1 \leqq x \leqq \pi$, respectively, where $1/100 > \delta_2 \geqq \delta_1 > 0$. Then

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \tag{4}$$

where

$$2\pi a_0 = \delta_1 + 2\delta_2, \qquad \pi a_n = (\delta_1 n^2)^{-1}(\cos n(\delta_1 + \delta_2) - \cos n\delta_1). \tag{5}$$

We note that

$$|a_n| \leqq 5 \min(1/(\delta_1 n^2), \delta_2). \tag{6}$$

LEMMA 1. *If p runs over all rational primes $\equiv 1 \pmod 4$, T is a large positive integer, then*

$$\sum_p \Lambda(p) e^{-p/T} = \tfrac{1}{2}T + o(T)$$

as $T \to \infty$, where $\Lambda(p) = \log p$.

We shall not prove Lemma 1 as it is a standard theorem on primes in an arithmetic progression. Let us note here that, if p is an odd rational prime, $p \equiv 1 \pmod 4$ holds if and only if p is the norm of a prime ideal in F.

2. In the following let (α) run over all integral ideals in the Gaussian field F. Define $\chi(\alpha, n) = (\alpha/\bar{\alpha})^{2n}$, where n is any rational integer. Then $\chi(\alpha, n) = \chi(i\alpha, n) = \chi(-i\alpha, n) = \chi(-\alpha, n)$, so that $\chi(\alpha, n)$ depends only upon the ideal (α) and upon n. Further, $\chi(\alpha\beta, n) = \chi(\alpha, n)\chi(\beta, n)$. If $L(s, n) = \sum \chi(\alpha, n)(N_{FR}(\alpha))^{-s}$, $\mathrm{Re}(s) > 1$, then $L(s, n)$ satisfies the following conditions:

(A) $$L(s, n) = \prod_q (1 - \chi(q, n)(N_{FR}q)^{-s})^{-1}, \qquad \mathrm{Re}(s) > 1,$$

where q runs over all prime ideals in F.

(B) $$\text{If} \quad R(s) = (\pi/2)^{-s/2}L(s, n)\Gamma(s + 2n),$$

then $R(s)$ is meromorphic in the entire S-plane (actually, regular in the entire S-plane if $n \neq 0$), and $R(s) = WR(1 - s)$, where $|W| = 1$. (See Hecke [2], [3] and [4] for proofs. Also see Landau [5] for a proof.)

We assume $L(s, n) \neq 0$, for $\mathrm{Re}\, s > \frac{1}{2}$ and all n; we shall call this assumption the Extended Riemann Hypothesis.

Let p run over all rational primes which are the norms of prime ideals in F (i. e., $p = a^2 + b^2$ for some positive rational integers a and b). Let $\exp(i\theta_p) = ((a + bi)(a - bi)^{-1})^2$.

LEMMA 2. *If T is a large positive number and a a positive rational integer, then*

$$\sum_p (\cos n\theta_p)\Lambda(p)e^{-p/T} = O(T^{\frac{1}{2}} \log(n + 3)).$$

The proof of this lemma will not be given. The proof for Dirichlet L-series was given by Ankeny (cf. [1]), and the difference between these two types of L-series has no real relevance to the proof. The results of A. Selberg on the zeros on the zeros of Dirichlet L-series can be extended to this new type of L-series, and Selberg's results were the essential tools for the author's result in [1].

3. We are now in a position to prove Theorems I and II. First,

$$g(\theta_p, \delta_1, \delta_2) = \sum_{n=0}^{\infty} a_n \cos n\theta_p, \tag{7}$$

where $g(x, \delta_1, \delta_2)$ was defined in Section **1**, and θ_p in Section **2**. Multiplying both sides of (7) by $\Lambda(p)e^{-p/T}$ and summing over p, we obtain

$$\begin{aligned}\sum_p g(\theta_p, \delta_1, \delta_2)\Lambda(p)e^{-p/T} &= a_0 \sum_p \Lambda(p)e^{-p/T} + \sum_{n=1}^{\infty} a_n \sum_p (\cos n\theta_p)\Lambda(p)e^{-p/T} \\ &= a_0 \sum_p \Lambda(p)e^{-p/T} + O\Big(\sum_{n=1}^{\infty} |\, a_n \,| T^{\frac{1}{2}} \log(n+3)\Big),\end{aligned} \tag{8}$$

by Lemma 2. Now

$$\begin{aligned}\sum_{n=1}^{\infty} |\, a_n \,| \log(n+3) &= \sum_{n \leqq (\delta_1\delta_2)^{-1/2}} |\, a_n \,| \log(n+3) + \sum_{n > (\delta_1\delta_2)^{-1/2}} |\, a_n \,| \log(n+3), \\ &\leqq 5\delta_2 \sum_{n \leqq (\delta_1\delta_2)^{-1/2}} \log(n+3) + 5/\delta_1 \sum_{n > (\delta_1\delta_2)^{-1/2}} \log(n+3)/n^2 \\ &\leqq 20\delta_2^{\frac{1}{2}}\delta_1^{-\frac{1}{2}} \log 1/\delta_1\delta_2.\end{aligned} \tag{9}$$

and by (6)

Hence, substituting (9) into (8),

$$\begin{aligned}\sum_p g(\theta_p, \delta_1, \delta_2)\Lambda(p)e^{-p/T} &= \tfrac{1}{2}\pi^{-1}(\delta_1 + 2\delta_2)\sum_p \Lambda(p)^{-p/T} + O(T^{\frac{1}{2}}\delta_2^{\frac{1}{2}}\delta_1^{-\frac{1}{2}} \log \delta_1\delta_2).\end{aligned} \tag{10}$$

The latter formula will yield the proof of both Theorems I, II and of (2) by certain selection of δ_1, δ_2.

If

$$\delta_1 = (\log T)^{1-\epsilon/2}/T^{\frac{1}{2}}, \qquad \delta_2 = (\log T)^{1+\epsilon}/T^{\frac{1}{2}} \tag{11}$$

where ϵ is some arbitrarily small number, then

$$\begin{aligned}\sum_p g(\theta_p, \delta_1, \delta_2)\Lambda(p)e^{-p/T} &= \delta_2 T/\pi + \tfrac{1}{2}\delta_1 T + o(\delta_2 T) + O(T^{\frac{1}{2}}(\log T)^{1+3/4\epsilon}) \\ &= 1/\pi T^{\frac{1}{2}}(\log T)^{1+\epsilon} + O(T^{\frac{1}{2}}(\log T)^{1+3/4\epsilon}) + o(T^{\frac{1}{2}}(\log T)^{1+\epsilon}).\end{aligned}$$

Let $f_1(\theta, T)$ be 1 if $|\, \theta \,| \leqq \delta_2$ and 0 if $\delta_2 \leqq x \leqq 2\pi - \delta_2$. Then

$$g(x, \delta_1, \delta_2 - \delta_1) \leqq f(x, \delta_2) \leqq g(x, \delta_1, \delta_2)$$

and, using the values (11) of δ_1, δ_2,

$$\sum_p f_1(\theta_p, (\log T)^{1+\epsilon}/(T^{\frac{1}{2}}))\Lambda(p)e^{-p/T} = T^{\frac{1}{2}}/(\pi)(\log T)^{1+\epsilon} + o(T^{\frac{1}{2}}(\log T)^{1+\epsilon}).$$

Similarly, we could extend the result of $f_1(\theta, (\log T)^{1+\epsilon}/(T^{\frac{1}{2}}), \lambda)$ defined in the introduction. This results by merely changing the definition of $f_2(\theta, \delta_1, \delta_2)$ by a translation on θ of length λ. We shall leave this modification to the reader.

We have now completed the proof of Theorem I. To prove Theorem II and (2), we proceed as follows: Let $f_2(\theta, T)$ be 1 if $|\theta| < c_1 T^{-\frac{1}{2}} \log T$, and f_2 be 0 otherwise, where c_1 is a constant that will be chosen later. Then, in (10), letting $\delta_1 = \delta_2 = c_2 T^{-\frac{1}{2}} \log T$ and using Lemma 1

$$\sum_p g(\theta_p, c_2 T^{-\frac{1}{2}} \log T, c_2 T^{-\frac{1}{2}} \log T) \Lambda(p) e^{-p/T} = 3\pi^{-1} c_2 T^{\frac{1}{2}} \log T + O(T^{\frac{1}{2}} \log T),$$

$> T^{\frac{1}{2}} \log T$, if c_2 is chosen sufficiently large. If $c_1 = 3c_2$, then

$$f_2(x, T) \geqq g(x, c_2 T^{-\frac{1}{2}} \log T, c_2 T^{-\frac{1}{2}} \log T),$$

so that

$$\sum_p f_2(\theta_p, T) \Lambda(p) e^{-p/T} \geqq \sum_p g(\theta_p, c_2 T^{-\frac{1}{2}} \log T, c_2 T^{-\frac{1}{2}} \log T) > T^{\frac{1}{2}} \log T. \tag{12}$$

From (12) we can deduce (2) provided we can show that

$$\sum_{p > c_4 T} f_2(\theta_p, T)(p) e^{-p/T} < \tfrac{1}{2} T^{\frac{1}{2}} \log T$$

for c_4 sufficiently large. To this end, define

$$H(U) = \sum_{p \leqq U} f_2(\theta_p, T) \Lambda(p). \tag{13}$$

We note that $f_2(x, U) \leqq g(x, c_1 T^{-\frac{1}{2}} \log T, c_1 T^{-\frac{1}{2}} \log T)$ for $U \geqq T$. Hence, if $\delta_1 = \delta_2 = c_1 T^{-\frac{1}{2}} \log T$ in (10), then

$$\begin{aligned} e^{-1} H(U) &= e^{-1} \sum_{p \leqq U} f_2(\theta_p, U) \Lambda(p) \leqq \sum_p f_2(\theta_p, U) \Lambda(p) e^{-p/U} \\ &\leqq \sum_p g(\theta_p, c_1 T^{-\frac{1}{2}} \log T, c_1 T^{-\frac{1}{2}} \log T) \Lambda(p) e^{-p/U} \\ &= (3c_1/2\pi) T^{-\frac{1}{2}} \log T \sum_p \Lambda(p) e^{-p/U} + O(U^{\frac{1}{2}} \log T) \\ &= (3c_1/2\pi) U T^{-\frac{1}{2}} \log T + O(U^{\frac{1}{2}} \log T) < c_5 U T^{-\frac{1}{2}} \log T. \end{aligned} \tag{14}$$

Hence, by partial summation,

$$\sum_{p \geqq c_4 T} f_2(\theta_p, T) \Lambda(p) e^{-p/T} = \sum_{n \geqq c_4 T} H(n)(e^{-n/T} - e^{-(n+1)/T}), \tag{15}$$

which, by (14), is

$$\begin{aligned} &= T^{-1} \int_{c_4 T}^{\infty} H(U) e^{-U/T} dU \leqq T^{-1} \int_{c_4 T}^{\infty} (c_5 T^{-\frac{1}{2}} \log T) U e^{-U/T} dU \\ &= c_5 (c_4 + 1) e^{-c_4} T^{\frac{1}{2}} \log T \end{aligned}$$

$< \frac{1}{2}T^{\frac{1}{2}} \log T$ if c_4 is sufficiently large. Hence, by (12),

$$(16) \quad \sum_{p<c_4T} f(\theta_p, T)(p)e^{-p/T} > T\log T - \sum_{p\geqq c_4T} f_2(\theta_p)\Lambda(p)e^{-p/T} > \tfrac{1}{2}T\log T.$$

Hence there exists a prime q such that $q < c_4T$ and

$$(17) \quad |\theta_q| < c_1T^{-\frac{1}{2}}\log T.$$

As $q = a^2 + b^2$, $a > b > 0$,

$$\exp \pm i\theta_q = ((a+bi)(a-bi)^{-1}), \exp i\theta_q = \cos\theta_q + i\sin\theta_q.$$

Hence, by (17), for T sufficiently large,

$$bq^{-\frac{1}{2}} = \sin\theta_q/2 \leqq c_6T^{-\frac{1}{2}}\log T \leqq c_7q^{-\frac{1}{2}}\log q,$$

and so $b = O(\log q)$. This implies that there are infinitely many primes satisfying (2); in fact, from (12) we can obtain a lower bound for their density.

4. One can proceed to generalize Theorems I and II to all quadratic imaginary extensions of the rationals. If the class number h of F is greater than one, we must define our characters for ideals that are not necessarily principal. In F the group of classes is a finite Abelian group. Therefore, the group has a finite number of generators, and so every ideal in F can uniquely be put in the form $= (\alpha)A_1^{i_1}A_2^{i_2}\cdots A_r^{i_r}$, where $0 \leqq i < h_1, \cdots,$ $0 \leqq i_r < h_r$, with $h = h_1h_2\cdots h_r$, and $A_1, \cdots, A_r$, are certain ideals in F. Also $A_i^{h_i} = (\beta_i)$ with $\beta_i \varepsilon F$.

Let $\beta_i^* = \beta_i^{1/h_i}$; this definition of β_i^* is unique up to an h_i-th root of unity. Define

$$\chi_j(A, n) = (\alpha/\bar{\alpha}\prod_{j=1}^{r}(\beta_j^*/\bar{\beta}_j^*))^{en}$$

where e is half the number of roots of unity in F. Although $\beta_i^*/\bar{\beta}_i^*$ is not uniquely determined, whatever choice of β_i^* we make, determines χ_j. As $h_1h_2\cdots h_r = h$, we have at most h different characters.

To establish our theorems we must also introduce the characters $\tau_0, \tau_1, \cdots,$ τ_{h-1} which form the character group of the ideal classes in F.

If we define $\exp(i\theta_{pj}) = \chi_j(P, 1)$, then we can proceed as before to show that the θ_{pj} are strongly equidistributed in the sense of Theorem I. Furthermore, using the τ's, we can show that the θ_{pj} are strongly equidistributed, where p runs over all prime ideals in a given class in F. We will state the interpretation of (2) in this case.

Let $H(x, y) = \alpha x^2 + \beta xy + \alpha y^2$ be a positive definite binary quadratic

form associated with an ideal class in F. (See Landau [6] for the relationship between binary quadratic forms and ideal classes in quadratic fields.)

THEOREM III. *There are infinitely many primes q such that*

$$q = H(a, b), \qquad b = O(\log q).$$

If F is a quadratic real extension of R, then we define our characters as follows: If $\alpha \varepsilon F$, $\alpha \neq 0$, let $\chi(\alpha, n) = \exp(\pi i\, n(\log \alpha)(\log \alpha)^{-1}(\log \epsilon)^{-1})$, where α is the conjugate of α, and ϵ is the fundamental unit in F.

The generalization for defining a character for an element to defining a character for an ideal is as follows: Let $A = (\alpha)A_1^{i_1}A_2^{i_2} \cdots A_r^{i_r}$, $A_j^{h_j} = (\beta_j)$. Define $\beta_j^* = (\beta_j)^{1/h_j}$, $\bar{\beta}_j^* = (\bar{\beta}_j)^{1/h_j}$, where both β_j^* and $\bar{\beta}_j^*$ are real. Then

$$\chi_{l_j}(A_j, n) = \exp \pi i\, n(\log \beta_j^*/\log \beta_j(\log \epsilon)^{-1} + 2l_j/h_j),$$

where l_j runs through a set of residues $(\mathrm{mod}\, h_j/(2, h_j))$.

The condition in Theorem III that $f(x, y)$ need be associated with an ideal class in a quadratic imaginary extension of the rationals can now be reduced to an association with an ideal class in any quadratic extension of the rationals.

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ON THE L^2-SPACE OF A LIE GROUP.*

By G. D. Mostow.[1]

1. Introduction. The purpose of this paper is to prove that the L^2-space of a Lie group G has a base of functions which are analytic throughout G. For a compact group this follows readily from the Peter-Weyl Theorem together with the theorem that a closed subgroup of an analytic group is analytic. In order to establish the theorem for a general Lie group G, it is proved that any Lie group is analytically the direct product of a compact subgroup K and a euclidean subspace E; that, moveover, left Haar measure on G is the product measure of Haar measure on K by ordinary euclidean measure on E (relative to suitable coordinates). The main result follows from this. It is also proved that any Lie group can be embedded analytically with non-vanishing Jacobian in euclidean space.

2. Definitions and Preliminaries. We employ the terms "analytic submanifold," "analytic isomorphism of an analytic manifold," "analytic subgroup" in the sense defined in [1] (cf. pp. 74, 85, 100, 107).

Definition. Let ϕ be an analytic mapping of A onto B. Let A_0, B_0 be analytic submanifolds of A, B respectively. A_0 and B_0 are called ϕ-related if and only if A_0 is a connected component of $\phi^{-1}(B_0)$.

Definition. Let G be an analytic group, let $A_1, \cdots, A_n$ be analytic submanifolds of G, and let θ denote the mapping $(a_1, a_2, \cdots, a_n) \to a_1 \cdot a_2 \cdot \cdots \cdot a_n$ of $A_1 \times A_2 \times x \cdots \times A_n$ into G. We say

a) $G = A_1 \cdot A_2 \cdot \cdots \cdot A_n$ if θ is onto;

b) $G = A_1 \cdot A_2 \cdot \cdots \cdot A_n$(UN) if θ is one-to-one onto;

c) $G = A_1 \cdot A_2 \cdot \cdots \cdot A_n$(TP) if θ is a homeomorphism onto;

d) $G = A_1 \cdot A_2 \cdot \cdots \cdot A_n$(AN) if θ is an analytic isomorphism onto.

LEMMA 1. *Let G, G^* be analytic groups, and assume*

1. $G^* = A_1^* \cdot A_2^* \cdot \cdots \cdot A_n^*$(UN);
2. $G = A_1 \cdot A_2 \cdot \cdots \cdot A_n$(AN);

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3. *Each A_i and A_i^* are ϕ-related, where ϕ is a locally* 1-1 *homomorphism from G onto G^** (or vice versa).

Then $G^* = A_1^* \cdot A_2^* \cdot \cdots \cdot A_n^*(\mathrm{AN})$.

Proof. This lemma will be proved only for the case that ϕ is a homomorphism of G onto G^*. In the reverse case, the proof is essentially the same. We show that there is an open set about any element $x^* \in G^*$ on which the mapping

$$\theta^*\colon a_1^* a_2^* \cdots a_n^* \to (a_1^*, a_2^*, \cdots, a_n^*),\ a_i^* \in A_i^*$$

is an analytic isomorphism. Let $x^* = b_1^* \cdot b_1^* \cdots b_n^*$, $b_i^* \in A_i^*$ $(i = 1, \cdots, n)$ and let $x \in G$ be ϕ-related to x^*. Then $x = b_1 \cdot b_2 \cdot \cdots \cdot b_n$, where $b_i \in A_i$ and b_i is ϕ-related to b_i^* $(i = 1, \cdots, n)$. Let V_i be an open neighborhood of b_i^* in A_i^* on which ϕ is one-to-one and let $U = V_1 \cdot V_2 \cdot \cdots \cdot V_n$. Let Φ denote the mapping $(a_1, a_2, \cdots, a_n) \to (\phi(a_1), \cdots, \phi(a_n))$ of $V_1 \times V_2 \times \cdots \times V_n$ onto $\phi(V_1) \times \cdots \times \phi(V_n)$. Since ϕ is an analytic isomorphism on U, Φ is an analytic isomorphism. Let ϕ^* denote the mapping of the open set $\phi(U)$ onto U which is the inverse of ϕ. Inasmuch as $\theta^* = \Phi\theta\phi^*$ (ϕ^* maps first), where θ is the mapping $a_1 a_2 \cdots a_n \to (a_1, a_2, \cdots, a_n)$ of G onto $A_1 \times A_2 \times \cdots \times A_n$, and inasmuch as each of ϕ^*, θ, Φ are analytic isomorphisms, θ^* is an analytic isomorphism on the open set $\phi(U)$. Since x^* was an arbitrary point of G^*, θ^* is an analytic isomorphism and $G^* = A_1^* \cdot A_2^* \cdot \cdots \cdot A_n^*(\mathrm{AN})$.

LEMMA 2. *Let G be an analytic group, and let A, B be analytic subgroups. If $G = A \cdot B(\mathrm{UN})$, then $G = A \cdot B(\mathrm{AN})$.*

Proof. Let $\mathfrak{G}$, $\mathfrak{A}$, $\mathfrak{B}$ denote the Lie algebras of G, A, B respectively, and let $X_1, \cdots, X_a$ and $Y_1, \cdots, Y_b$ be bases for $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Then $X_1, X_2, \cdots, X_a, Y_1, \cdots, Y_b$ form a base for $\mathfrak{G}$. Let

$$U = \{z \mid z = \exp s^1 X_1 \cdot \cdots \exp s^a X_a, \mid s^i \mid < \epsilon,\ i = 1, \cdots, a\},$$

$$V = \{z \mid z = \exp t^1 Y_1 \cdot \cdots \exp t^b Y_b, \mid t^i \mid < \epsilon,\ i = 1, \cdots, b\}.$$

Then for ϵ suitably small, U, V, and UV are open neighborhoods of the identity in A, B, and G respectively and the mapping $\phi\colon (u, v) \to u \cdot v$ is an analytic isomorphism of $U \times V$ onto $U \cdot V$ (cf. [5], p. 193). Select ϵ so that these conditions hold.

Let L_x and R_x for $x \in G$ denote respectively left and right group translation by x. Let L_a^* for $a \in A$ and R_b^* for $b \in B$ denote the transformations $(x, y) \to (ax, y)$ and $(x, y) \to (x, yb)$ of $A \times B$ onto itself. Since A, B,

and G are analytic groups, L_x, R_x, L_a^*, and R_b^* are analytic isomorphisms. Let (x, y) be any element of $A \times B$ and let θ denote the mapping $(a, b) \to ab$ of $A \times B$ onto G. On the open set (xU, Vy) we have $\theta = L_x R_y \phi L_x^{*-1} R_y^{*-1}$, so that θ is an analytic isomorphism. Thus θ is a regular analytic mapping at all points of $A \times B$ and hence is an analytic isomorphism.

As a consequence of Lemma 2, it is readily seen that the mapping $ab \to a_0 a \cdot b_0 b$ of G onto G is an analytic isomorphism.

3. Decomposition of Haar measure.

Definition. A measure on an analytic n-dimensional manifold is an alternating analytic n-form (cf. [1], p. 146).

Definition. A left Haar measure on an analytic group is a non-zero left invariant measure (cf. [1], p. 167).

Suppose that $G = A \cdot B(\mathrm{AN})$. In the sequel we let α and β denote the mappings $a \cdot b \to a$ of G onto A and $a \cdot b \to b$ of G onto B respectively. Suppose that w_A and w_B are measures on A and B respectively.

Definition. $w_A \times w_B = \delta\beta w_A \square \delta\beta w_B$ (cf. [1] p. 151 for meaning of notation). This is called "the product measure."

As is known (cf. [1], p. 166),

$$\int_{A\times B} h w_A \times w_B = \int_B \left(\int_A h w_A\right) w_B$$

for any function h on $A \times B$. If $G = A_1 \cdot A_2 \cdot \cdots \cdot A_n(\mathrm{AN})$ and $w_1, \cdots, w_n$ are measures on $A_1, \cdots, A_n$, we define $w_1 \times w_2 \times \cdots \times w_n$ analogously.

Let $L(g)$ and $R(g)$ denote left and right group translation by g respectively. If $G = A \cdot B(\mathrm{AN})$ where A and B are subgroups, let $L^*(a_0 b_0)$ denote the analytic isomorphism $ab \to a_0 a \cdot b_0 b$. Let $L'(a)$ denote restriction of $L(a)$ to A if $a \in A$, and let $L'(b)$ denote the restriction of $L(b)$ to B if $b \in B$.

LEMMA 3. *Suppose that A and B are subgroups of the analytic group G such that $G = A \cdot B(\mathrm{AN})$. Let ϕ and ψ denote left Haar measures on A and B respectively. Then $(\phi \times \psi)_{ab} = \delta L^*(a^{-1}b^{-1})_{ab}(\delta\alpha_e \phi_e \square \delta\beta_e \psi_e)$, where e is the identity element of G.*

Proof. By definition, $(\phi \times \psi)_{ab} = \delta\alpha_{ab}\phi_a \square \delta\beta_{ab}\psi_b$. Since ϕ and ψ are left Haar measures, $\phi_a = \delta L'(a^{-1})_a \phi_e$ and $\psi_b = \delta L'(b^{-1})_b \psi_e$. Since

$$L'(a^{-1})\alpha = \alpha L^*(a^{-1}b^{-1}), \qquad \delta L'(a^{-1}) = \delta L^*(a^{-1}b^{-1})\delta\alpha.$$

Consequently

$$\delta\alpha_{ab}\phi_a = \delta\alpha_{ab}\delta L'(a^{-1})_{a}\phi_e = \delta L^*(a^{-1}b^{-1})_{ab}\delta\alpha_e\phi_e.$$

Similarly $\delta\beta_{ab}\psi_b = \delta L^*(a^{-1}b^{-1})_{ab}\delta\beta_e\psi_e$. It follows that

$$\begin{aligned}(\phi \times \psi)_{ab} &= \delta L^*(a^{-1}b^{-1})_{ab}\delta\alpha_e\phi_e \square \delta L^*(a^{-1}b^{-1})_{ab}\delta\beta_e\psi_e \\ &= \delta L^*(a^{-1}b^{-1})_{ab}(\delta\alpha_e\phi_e \square \delta\beta_e\psi_e).\end{aligned}$$

Let $\mathfrak{G}$ denote the Lie algebra of G, and let $\mathfrak{A}$, $\mathfrak{B}$ denote the Lie subalgebras of A and B respectively.

LEMMA 4. *Let w denote the left Haar measure determined by*

$$\delta\alpha_e\phi_e \square \delta\beta_e\psi_e, \text{ i. e., } w_{ab} = \delta L(b^{-1}a^{-1})_{ab}(\delta\alpha_e\phi_e \square \delta\beta_e\psi_e)$$

Let $T(b)$ denote the linear transformation induced on the factor space $\mathfrak{G}/\mathfrak{B}$ by the differential at e of the inner automorphism $L(b^{-1})R(b)$, where $b \in B$. Then $w = f \cdot \phi \times \psi$, where f is the function on G with

$$f(a \cdot b) = \det T(b) \equiv |T(b)|.$$

Proof. It follows directly from Lemma 3 that

$$\begin{aligned}w_{ab} &= \delta L(b^{-1}a^{-1})_{ab}\delta L^*(ab)_e(\phi \times \psi)_{ab} \\ &= |\delta L(b^{-1}a^{-1})_{ab}\delta L^*(ab)_e|\,(\phi \times \psi)_{ab} \\ &= |dL^*(ab)_e dL(b^{-1}a^{-1})_{ab}|\,(\phi \times \psi)_{ab} \\ &= |dL(b^{-1}a^{-1})_{ab}dL^*(ab)_e|\,(\phi \times \psi)_{ab}.\end{aligned}$$

To evaluate the determinant of the linear transformation $dL(b^{-1}a^{-1})dL^*(ab)$ on $\mathfrak{G}$, we select as a base for $\mathfrak{G}$ the union of a base for $\mathfrak{A}$ and a base for $\mathfrak{B}$.

On A, $L(b^{-1}a^{-1}) \cdot L^*(ab) = L(b^{-1})R(b)$, and on B, $L(b^{-1}a^{-1}) \cdot L^*(ab)$ is the identity transformation. Consequently

$$f(a,b) \equiv |dL(b^{-1}a^{-1})_{ab}dL^*(ab)_e| = \begin{vmatrix} T_m(b) & J \\ 0 & I \end{vmatrix} = |T_m(b)|,$$

where $T_m(b)$ is the matrix form of $T(b)$ relative to suitable choice of base in $\mathfrak{G}/\mathfrak{B}$, and I is an identity matrix. It follows that $f(a,b) = |T(b)|$.

COROLLARY 4·1. *If B is normal, then $f(a,b) = 1$.*

Proof. If $X \in \mathfrak{A}$ and $Y \in \mathfrak{B}$, then

$$dL(\exp -Y)dR(\exp Y)X = X[I + adY + (adX)^2/2! + \cdots] = X + Z,$$

where $Z \in \mathfrak{B}$. Since any element of B is a finite product of exponentials,

$dL(b^{-1})dR(b)(X) = X + Z$, where Z is some element of $\mathfrak{B}$ for all $b \in B$. It follows that $T(b)$ is the identity transformation for all $b \in B$, and so $f(a, b) = |T(b)| = 1$.

COROLLARY 4 · 2. *If B is compact or semi-simple, then $f(a, b) = 1$.*

Proof. $|T(b)|$ is a real one-dimensional representation of the compact (or semi-simple) connected group B. It follows that $|T(b)| = 1$ for all $b \in B$.

COROLLARY 4 · 3. *If the analytic subgroup determined by the Lie algebra $\mathfrak{B}$ in some analytic group with Lie algebra $\mathfrak{G}$ is compact, then $|T(b)| = 1$ for all $b \in B$.*

Proof. Let G' be an analytic group with Lie algebra $\mathfrak{G}$, let B' be the analytic subgroup determined by $\mathfrak{B}$, and assume B' is compact. Let π denote the local isomorphism between G and G'. Then $T(b) = T(\pi(b))$ when $b \in B$. Consequently, $|T(b)| = 1$ for all b in some neighborhood U of the identity in B. Since T is a homomorphism of B, $|T(b_1 b_2 \cdots b_n)| = 1$ when $b_1 \cdots b_n \in U$. Since B is connected, $\Sigma U^n = B$. Hence $|T(b)| = 1$ for all $b \in B$.

COROLLARY 4 · 4. *If* ad Y *is nilpotent for all $Y \in \mathfrak{B}$, then $f(a, b) = 1$.*

Proof. If $Y \in B$, then $T(\exp Y)$ is the exponential of a nilpotent transformation and therefore has determinant one. It follows that $|T(b)| = 1$ for all $b \in B$.

LEMMA 5. *Suppose that G and G^* are analytic groups and that π is a locally one-to-one homomorphism of G onto G^*. Assume that $G = M \cdot N(\mathrm{AN})$, $G^* = M^* \cdot N^*(\mathrm{AN})$, and that M, M^* and N, N^* are π-related. Suppose that w_M, w_N, w_G, w_{M^*}, w_{N^*}, w_{G^*} are measures on M, N, G, M^*, N^*, G^* respectively such that*

$$\delta\pi\delta\mu^* w_{M^*} = \delta\mu w_M, \qquad \delta\pi\delta\nu^* w_{M^*} = \delta\nu w_N, \qquad \delta\pi w_{G^*} = w_G,$$

where μ, ν and μ^*, ν^* denote the projections of G onto M, N and of G^* onto M^*, N^* respectively. Then $w_{G^*} = w_{M^*} \times w_{N^*}$ if and only if $w_G = w_M \times w_N$.

The proof follows directly from the fact that $d\pi$ is a regular onto mapping at each point and hence that $\delta\pi$ is an isomorphism of the ring of differential forms. Suppose G is an analytic group with measure w and $A_1, \cdots, A_n$ are submanifolds with measures $w_1, \cdots, w_n$.

Definition. $G = A_1 \cdot A_2 \cdots A_n$ $(w, w_1, \cdots, w_n)$ means

$$G = A_1 \cdot A_2 \cdots A_n(\mathrm{AN}) \text{ and } w = w_1 \times w_2 \cdots \times w_n.$$

If $A_1, A_2, \cdots, A_n$ are analytic subgroups, then $G = A_1 \cdot A_2 \cdots A_n(\mathrm{m})$ means $G = A_1 A_2 \cdots A_n(\mathrm{AN})$ and left Haar measure on G is the product of left Haar measures on $A_1, A_2, \cdots A_n$.

If G and G^* are analytic groups, if w^* is a left Haar measure on G^*, and if π is a locally one-to-one homomorphism of G onto G^*, then $\delta_\pi w^*$ is a left Haar measure on G. For $L^{-1}(g^*)\pi = \pi L^{-1}(g)$, where $g^* = \pi(g)$, implies

$$\delta\pi_g w^*_{g^*} = \delta\pi_g \delta L^{-1}{}_{g^*}(g^*)\, w^*_{e^*} = \delta L^{-1}{}_g(g)\,(\delta\pi_e w^*_{e^*}),$$

where e, e^* are the identity elements in G, G^* respectively. Thus $\delta\pi w^*$ is the left Haar measure determined by $\delta\pi_e w^*_{g^*}$. It follows that if w is any Haar measure on G, then w and $\delta\pi w^*$ differ only by a constant factor.

THEOREM 1. *Let G be an analytic group. Then $G = E \cdot K(w_G, w_E, w_K)$, where K is a compact subgroup, E is analytically isomorphic to a euclidean space, w_G and w_K are left Haar measures, and w_E is euclidean measure on E relative to suitable coordinates.*

Proof. We shall make use of topological decompositions due to Malcev [2] and Iwassawa [3].

Let $\tilde{G}$ denote the simply connected group of G, let π denote the natural homomorphism of $\tilde{G}$ onto G, and let $\tilde{L} \cdot \tilde{R}$ be a Levi decomposition of $\tilde{G}$ into the semi-direct product of a semi-simple subgroup $\tilde{L}$ and the radical $\tilde{R}$. Let $L = \pi(\tilde{L})$, let $L^* = \mathrm{Ad}(L)$, where Ad denotes the adjoint representation. Throughout our discussion, analytic subgroups of L^* and L that are denoted by X^* and X are Ad-related; analytic subgroups of $\tilde{G}$ and G that are denoted by $\tilde{X}$ and X are π-related.

By a theorem of Iwassawa [2] which is related to the representation of a matrix as the product of a triangular by an orthogonal matrix, $L^* = S^* \cdot C^*(\mathrm{TP})$ where S^* is a simply connected, solvable analytic subgroup and C^* is a compact analytic subgroup (cf. also [4]). From this one obtains readily that $L = S \cdot C(\mathrm{TP})$, $C = A \cdot K_1(\mathrm{TP})$ where K_1 is a compact subgroup and A is a vector subgroup central in C, $L = S \cdot C(\mathrm{TP})$, and the center of L is contained in C. (This last result follows from the fact that L/C is a covering space of the simply connected space $S^* = L^*/C^*$ and is therefore univalent.) Thus $\tilde{G} = \tilde{L} \cdot \tilde{R} = \tilde{S} \cdot \tilde{C} \cdot \tilde{R} = \tilde{S}(\tilde{C}\tilde{R})(\mathrm{TP})$, where $\tilde{C}\tilde{R}$ contains the center of $\tilde{G}$. It follows directly that $G = S \cdot (CR)(\mathrm{TP})$.

Now, by a theorem of Malcev, $CR = U_1 \cdot U_2 \cdot \cdots \cdot U_n \cdot (K_3 K_2 K_1)(\mathrm{TP})$, where 1) U_i is a subgroup isomorphic to the reals; 2) $U_i \cdots U_n \cdot K_3$ is a subgroup $(i = 1, \cdots, n)$ that is normal in $U_{i-1} \cdots U_n \cdot K_3$ $(i = 2, \cdots, n)$,

3) K_3, K_2 are compact subgroups of the solvable group AR which commute with each other and with K_1 (K_3 is in the closure of the commutator subgroup of AR); 4) $\tilde{K}_3\tilde{K}_2\tilde{K}_1 = \tilde{K}_3 \cdot \tilde{K}_2 \cdot K_1(\mathrm{TP})$. Thus

$$\begin{aligned}
\tilde{C}\tilde{R} &= (\tilde{U}_1\tilde{U}_2 \cdot \cdot \cdot \tilde{U}_n\tilde{K}_3) \cdot (\tilde{K}_2\tilde{K}_1)(\mathrm{TP}) \\
&= (\tilde{U}_1\tilde{U}_2 \cdot \cdot \cdot \tilde{U}_n\tilde{K}_3) \cdot (\tilde{K}_2\tilde{K}_1)(\mathrm{AN}) && \text{(Lemma 2)} \\
&= (\tilde{U}_1\tilde{U}_2 \cdot \cdot \cdot \tilde{U}_n\tilde{K}_3) \cdot (\tilde{K}_2\tilde{K}_1)(\mathrm{m}) && \text{(Corollary 4 · 3)} \\
&= [\tilde{U}_1 \cdot (\tilde{U}_2 \cdot \cdot \cdot \tilde{U}_n\tilde{K}_3)] \cdot (\tilde{K}_2\tilde{K}_1)(\mathrm{m}) && \text{(Corollary 4 · 1)} \\
&= [\tilde{U}_1 \cdot \tilde{U}_2 \cdot \cdot \cdot \tilde{U}_n \cdot \tilde{K}_3](\tilde{K}_2\tilde{K}_1)(\mathrm{m}) && \text{(Corollary 4 · 1)} \\
&= \tilde{U}_1 \cdot \tilde{U}_2 \cdot \cdot \cdot \tilde{U}_n \cdot (\tilde{K}_3 \cdot \tilde{K}_2\tilde{K}_1)(\mathrm{m})
\end{aligned}$$

since the formation of the direct product of measure spaces is associative. But $\tilde{K}_3 \cdot \tilde{K}_2\tilde{K}_1 = \tilde{K}_3\tilde{K}_2\tilde{K}_1(\mathrm{m})$ by Corollary 4 · 3. Hence

$$\tilde{C}\tilde{R} = \tilde{U}_1 \cdot \tilde{U}_2 \cdot \cdot \cdot \tilde{U}_n \cdot (\tilde{K}_3\tilde{K}_2\tilde{K}_1)(\mathrm{m})$$

$$\text{and } CR = U_1, U_2, \cdot \cdot \cdot, U_n \cdot (K_3K_2K_1)(\mathrm{m})$$

by Lemmas 1 and 5. But $\tilde{G} = \tilde{L} \cdot \tilde{R}(\mathrm{m}) = (\tilde{S} \cdot \tilde{C}) \cdot \tilde{R}(\mathrm{m})$, by Corollary 4 · 1 and Corollary 4 · 3, hence $\tilde{G} = \tilde{S} \cdot (\tilde{C} \cdot \tilde{R})(\mathrm{m})$. Consequently, $G = S \cdot (CR)(\mathrm{m})$ by Lemmas 1 and 5.

As is well known for simply connected solvable groups, $S = V_1 \cdot V_2 \cdot \cdot \cdot V_m$, where each V_i is isomorphic to the reals and $V_{i+1} \cdot \cdot \cdot V_m$ is normal in $V_i \cdot \cdot \cdot V_n$ $(i = 1, \cdot \cdot \cdot, n)$. Hence, by repeated use of Corollary 4 · 1, we obtain

$$G = V_1 \cdot V_2 \cdot \cdot \cdot V_m \cdot U_1 \cdot \cdot \cdot U_n \cdot (K_3K_2K_1)(\mathrm{m}).$$

Let

$E = V_1V_2 \cdot \cdot \cdot V_mU_1 \cdot \cdot \cdot U_n$, $K = K_3K_2K_1$, and let $w_E = w_1 \times \cdot \cdot \cdot \times w_{m+n}$,

where w_i is left Haar measure in the i-th factor. Let $\mathfrak{R}_i$ denote the Lie subalgebra of the i-th factor $(i = 1, \cdot \cdot \cdot, m + n)$, let $\mathfrak{R}^{m+n}$ denote the Cartesian product $\mathfrak{R}_1 \times \cdot \cdot \cdot \times \mathfrak{R}_{m+n}$, and let θ denote the analytic isomorphism

$$(a_1, \cdot \cdot \cdot, a_{m+n}) \to \exp a_1, \cdot \exp a_2 \cdot \cdot \cdot \exp a_n, \qquad a_i \in \mathfrak{R}_i, \quad (i = 1, \cdot \cdot \cdot, m + n),$$

of $\mathfrak{R}^{m+n}$ onto E and let π_i denote the projection of $\mathfrak{R}^{m+n}$ onto $\mathfrak{R}_i$. Then

$$\delta\theta(w_1 \times w_2 \times \cdot \cdot \cdot \times w_{m+n}) = \delta\pi_1(u_1) \times \cdot \cdot \cdot \times \delta\pi_{m+n}(u_{m+n}),$$

where $u_i = \delta \exp(w_i)$ $(i = 1, \cdot \cdot \cdot, m + n)$. But, as is well known, $\delta \exp(w_i)$ is ordinary euclidean measure on the one (dimensional) linear space $\mathfrak{R}_i$ (cf.

[1], p. 155). Thus $\delta\theta(w_1 \times \cdots \times w_{m+n})$ is euclidean measure on $\mathfrak{R}^{m+n}$. Hence $G = E \cdot K(w_G; w_E, w_K)$ where w_G, w_K are left Haar measures, and w_E becomes ordinary euclidean measure when the coordinate system θ^{-1} is introduced on the analytic manifold E.

4. The theorems on embedding and on a base of analytic functions.

THEOREM 2. *A Lie group can be embedded analytically (with non-vanishing Jacobian) in Euclidean space.*

Proof. The Peter-Weyl Theorem tells us that a compact Lie group admits a (continuous) faithful finite dimensional representation. Since a continuous isomorphism of a Lie group is an analytic isomorphism ([1], p. 128), and since an analytic subgroup of the group of $n \times n$ matrices is an analytic submanifold of the euclidean space of all $n \times n$ matrices ([1], p. 101), a continuous faithful finite dimensional representation of a Lie group provides an analytic embedding with non-vanishing Jacobian in euclidean space. Consequently any analytic group G can be analytically embedded in euclidean space. For if $G = E \cdot K(\mathrm{AN})$ where K is compact and E is euclidean, and if θ is a faithful n-dimensional representation of K, then the mapping $\Theta:(e,k)\to(e,\theta(k))$ of G into the euclidean space $E \times E^{n^2}$ is an analytic isomorphism of G. Finally if G' is a Lie group, and if G_0 denotes the component of the identity of G, and if $\{G_n \mid n = 1, 2, \cdots\}$ denote the other connected components of G', then G' can be embedded analytically in Euclidean space as follows: Let θ be any analytic embedding of G_0 in the euclidean space E. Select for each G_i an element x_i $(i = 1, 2, \cdots)$. Define $\theta'(x_n g) = (\theta(g), n)$ for all $g \in G_0$ and for all n. Then θ' is an analytic embedding of G into the euclidean space $E_0 \times R$, where R denotes the real numbers.

THEOREM 3. *The L^2 space of a Lie group G has a base of functions which are analytic on G.*

Proof. We may obviously assume that G is connected. Thus $G = E \cdot K$ (W_G, W_E, W_K), where K is a compact subgroup, E is mapped by an analytic isomorphism θ onto a euclidean space R, and $\delta\theta(W_E)$ is ordinary euclidean measure on R. Let ϕ denote a faithful representation of the compact group K by matrices of degree n, let F denote the euclidean space of all $n \times n$ matrices (f_{ij}), the f_{ij} being regarded as coordinate functions $(i, j = 1, 2, \cdots, n)$. Let f'_{ij} denote the function $f_{ij}\phi$ and let $f'_1, f'_2, \cdots, f'_n, \cdots$ be an ortho-

normal base with respect to Haar measure on K of the ring of functions generated by $\{f'_{ij} \mid i, j = 1, \cdots, n\}$. Inasmuch as ϕ is an analytic *isomorphism* into F (i. e. with non-vanishing Jacobian), $f'_1, f'_2, \cdots$ are analytic on K and form, in fact, a base for the L^2 space of K (with respect to Haar measure). Let $\{g'_1, g'_2, \cdots, g'_n, \cdots\}$ be a base of analytic functions for the L^2 space of the euclidean space R. Let α, β denote the projections of G onto E, K respectively, and let $f_i = f'_i\beta$, $g_i = g'_i\theta\alpha$. Then $\{f_i \cdot g_j \mid i, j = 1, 2, \cdots\}$ is a base of analytic functions for the L^2-space of G.

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LOCALLY COMPACT RINGS. III.*

By IRVING KAPLANSKY.

1. Introduction. In a series of papers ([3], [4], [5]) the author has carried out various investigations on locally compact rings. In the present paper we shall add three further theorems, as follows.

(a) Two earlier results ([4], theorems 13 and 14) are generalized by showing that a non-discrete locally compact primitive ring of characteristic 0 is a finite-dimensional algebra.

(b) A conjecture ([4], p. 459) is verified by showing that a non-discrete locally compact simple[1] ring with minimal ideals is a finite-dimensional algebra.

(c) A structure theory is given for locally compact right bounded semi-simple rings.

2. Primitive rings of characteristic 0. The theorem to be proved in this section is the following:

THEOREM 1. *Let A be a non-discrete locally compact primitive ring of characteristic 0. Then A is a finite-dimensional algebra over its center.*

We need three lemmas. The first two are implicit in [4], Lemma 10 and Theorem 11, and are given without further proof.

LEMMA 1. *Let A be a locally compact non-discrete totally disconnected ring. Then for some prime p, the set of all x with $p^n x \to 0$ forms a non-zero closed ideal in A.*

LEMMA 2. *Let A be a locally compact ring which is not a Q-ring. Then there exists a non-zero idempotent e in A such that eAe is a Q-ring.*

The next lemma is a recapitulation of Jacobson's theory ([1], p. 236) of the eigenring; we drop the assumption of a unit element and insert topological trimmings.

* Received January 12, 1952.

[1] By a simple ring we shall always mean one that has no proper two-sided ideals, and not merely no proper closed two-sided ideals.

LEMMA 3. *Let A be a topological ring and M a closed regular maximal right ideal in A. Let B be the set of all x in A with $xM \subset M$. Then B is a closed subring of A, M is a two-sided ideal in B, B/M is a topological division ring, and A/M is a topological linear space over B/M.*

Proof. Let e be a left unit mod M. The ring A acts by right multiplication on A/M to produce an irreducible ring of endomorphisms, say with commuting division ring D. Take θ in D and suppose θ sends the coset $e+M$ into $b+M$. Then since θ commutes with the right multiplication by x, we find that θ sends $ex+M$ into $bx+M$. Since $ex-x \in M$, we have $ex+M=x+M$. Thus θ sends $x+M$ into $bx+M$, and coincides with left multiplication by b. Necessarily $bM \subset M$, that is, $b \in B$. In this way we obtain a homomorphism of B onto D. The kernel K consists of all y in B with $yA \subset M$. Evidently K is a right ideal containing M. If $K \neq M$, then $K=A$, $A^2 \subset M$, which is impossible since e^2 is not in M. Thus $K=M$ and $B/M \simeq D$. The remaining statements of the lemma are routine, and are left to the reader.

Proof of Theorem 1. *Case* I. A is a Q-ring and primary (in the sense that for some prime p, $p^n x \to 0$ for every x). Take a regular maximal right ideal M such that A is faithfully represented on A/M. Since A is a Q-ring, M is closed. So we are able to apply Lemma 3. The vital thing is to know that the locally compact division ring B/M is non-discrete; this follows from the assumption that A, and hence B/M, is primary. By [4], Lemma 9, A/M is finite-dimensional over B/M. From this it follows that A is a simple algebra finite-dimensional over its center.

Case II. A is primary but not a Q-ring. We apply Lemma 2 and obtain a non-zero idempotent e such that eAe is a Q-ring. Moreover eAe is again primitive, and again primary. By Case I, eAe is a finite-dimensional algebra. From this it follows that A at least has minimal ideals (indeed if f is a primitive idempotent in eAe, it is also a primitive idempotent in A). This allows us to quote [4], Theorem 14 to complete the proof of Case II.

Case III. A is arbitrary. By [4], Theorem 2 we can assume that A is totally disconnected. We cite Lemma 1, and find a non-zero closed primary ideal I in A. Also I is again primitive and again locally compact. On applying Case I or II, which ever is appropriate, we deduce that I is a finite-dimensional algebra. In particular it has a unit element, whence it is a direct sumamnd of A. This is incompatible with the primitivity of A, unless $I=A$. This concludes the proof of Theorem 1.

3. Simple rings with minimal ideals. With Theorem 1 we have concluded the study of locally compact primitive rings of characteristic 0. But characteristic p is another matter; the example on page 459 of [4] is an indication of the numerous possibilities. But if we go so far as to assume a simple ring with minimal ideals, we get the usual strong result.

THEOREM 2. *A non-discrete locally compact simple ring with minimal ideals is a finite-dimensional algebra over its center.*

Two lemmas are needed, the first purely algebraic.

LEMMA 4. *Let A be a simple ring with minimal ideals, and $\{e_i\}$ an infinite set of orthogonal idempotents in A. Then there cannot exist in A an element y such that, for every i, e_iy is a non-zero element of e_iAe_i.*

Proof. Represent A as a ring of finite-valued linear transformations on a vector space. Then the proposed element y, as a linear transformation, would have infinite-dimensional range. For its range contains the range (say S_i) of $e_iy = e_iye_i$; and each S_i is non-zero and is disjoint from the union of the remaining S_j's.

LEMMA 5. *Let A be a simple non-discrete topological ring, U a neighborhood of 0 in A, e an idempotent such that $U \cap (1-e)A(1-e) = 0$. Then e is the unit element of A.*

Proof. Suppose $e \neq 1$. By the simplicity of A we have $A = A(1-e)A$, whence

$$e = \sum_{i=1}^{n} x_i(1-e)y_i. \tag{1}$$

There exists a neighborhood V of 0 such that $V \subset U$ and

$$(1-e)V(1-e) \subset U, \tag{2}$$

$$(1-e)Vx_j(1-e) \subset U, \tag{3}$$

$$(1-e)y_iV(1-e) \subset U, \tag{4}$$

$$(1-e)y_iVx_j(1-e) \subset U. \tag{5}$$

Since $U \cap (1-e)A(1-e) = 0$, the left side of (2)-(5) is in each case 0. Multiply (3) on the right by y_j, add for $j = 1, \cdots, n$ and use (1); we get $(1-e)Ve = 0$. Similarly from (4) and (5) we find $eV(1-e) = 0$, $eVe = 0$. On adding these four equations we obtain $V = 0$, contradicting the non-discreteness of A.

Proof of Theorem 2. It is sufficient to prove that A has a unit element, for then it has the descending chain condition, etc. Suppose the contrary and let U be be a compact open subring in A. Take any non-zero element x_1 in U.

It is known ([2], Theorem 9) that $x_1 \in e_1 A e_1$ for a suitable idempotent e_1. By Lemma 5, we can pick a non-zero element x_2 in $U \cap (1 - e_1) A (1 - e_1)$; again $x_2 \in e_2 A e_2$, where e_2 is a suitable idempotent orthogonal to e_1. Continuing in this fashion, we construct sequences x_i, e_i with $x_i \neq 0$, $\{e_i\}$ orthogonal idempotents, $x_i \in U \cap e_i A e_i$. Write $y_i = x_1 + \cdots + x_i$; then $y_i \in U$. By compactness there exists in U a limit point y of the sequence $\{y_i\}$. We have $e_m y_n = x_m$ for all $n \geqq m$, and hence $e_m y = x_m$. We have contradicted Lemma 4.

4. Right bounded rings. In [4], Theorem 4 it was shown that a locally compact bounded semi-simple ring is a direct sum of a compact ring and a discrete ring. We shall now obtain a fairly complete structure theorem under the weaker hypothesis of right boundedness.

THEOREM 3. *Let A be a locally compact right bounded semi-simple ring. Then A contains an open ideal which is a local direct sum of finite simple rings relative to non-zero right ideals.*

LEMMA 6. *Let A be a semi-simple ring and I a right ideal in A. Then the radical $R(I)$ of I is its left annihilator. If $I/R(I)$ has a unit element, then I is of the form $I = eA$, e an idempotent.*

Proof. Let $x \in R(I)$, $y \in I$. To prove $xy = 0$ it is enough to prove that xy is in the radical of A; that is, we must show that xyz is quasi-regular for every z in A. But $x(yz) \in xI$ is in $R(I)$, and so is even quasi-regular in I.

Suppose further that $I/R(I)$ has a unit element, and let $e \in I$ be any element mapping on it. Then $(1 - e)I \subset R(I)$, whence $[(1 - e)I]^2 = 0$, $(1 - e)I$ is a nilpotent right ideal in A, $(1 - e)I = 0$. This proves that e is an idempotent and $I = eI \subset eA$. Further $I \supset eA$ and so $I = eA$.

LEMMA 7. *Let A be a semi-simple topological ring, and g a primitive idempotent in A. Then the closure of AgA is a primitive ring.*

Proof. Let C be the closure of AgA. We claim that the representation of C upon the right ideal $gC = gA$ is faithful. If not, there exists a non-zero element x in C with $gAx = 0$. But then $Cx = 0$, which contradicts the fact that C, an ideal in a semi-simple ring, is itself semi-simple.

LEMMA 8. *Let A be a locally compact right bounded semi-simple ring. Then A has a compact open right ideal of the form eA, with e an idempotent. Suppose further that P is a closed primitive ideal in A not containing e; then A/P is finite, and P is a direct summand of A.*

LEMMA 9. *A locally compact right bounded primitive ring is discrete.*

Proof. It is convenient to prove these two lemmas together. Let A be as in Lemma 8. An application of [4], Theorem 1 shows that A must be totally disconnected. Then A has compact open subgroups. By [3], Lemma 9 it even has a compact open right ideal I. Let $R(I)$ be the radical of I; $R(I)$ is closed by [5], Theorem 1. Then by [3], Theorem 16 $I/R(I)$ has a unit element. This makes Lemma 6 applicable and shows that I is of the form eA. We have proved the first part of Lemma 8.

We now suppose further that P is a closed primitive ideal in A not containing e. Write $B = A/P$, and let f be the image of e in B; note that f is non-zero. We have that fB is compact open, since it is the image of eA. Likewise fBf is compact, and moreover fBf (like B) is primitive. But a compact primitive ring is necessarily finite ([3] remark 7 on page 165). So fBf is finite, and this proves that B is a primitive ring with minimal ideals. Indeed f can be written as the sum of a finite number of idempotents which are primitive idempotents in fBf and likewise in B. Hence f lies in the unique minimal two-sided ideal of B, which we shall call D. Since D contains fB, it is open in B. Thus D is a locally compact simple ring with minimal ideals. If D is non-discrete, then by Theorem 2 it is a finite-dimensional algebra over a non-discrete center; but this is incompatible with the fact that D contains a compact open right ideal $fB = fD$. Hence D is discrete, whence fB is finite, from which it follows that B itself is finite (for an infinite primitive ring cannot have a finite non-zero right ideal).

We have now completed the proof of Lemma 8, except for the fact that P is a direct sumand of A. Before showing this, we pause to prove Lemma 9. We accordingly let A be a locally compact right bounded primitive ring and let eA be a compact open right ideal, as above. There are two possibilities. If $e = 0$, then A is of course discrete. If $e \neq 0$, we may take $P = 0$ in the above discussion, and we conclude that A is even finite.

We are now ready to conclude the proof of Lemma 8 by showing that P is a direct summand of A. For this purpose we look next at the compact semi-simple ring eAe, whose structure is completely known from [3], Theorem 16; in particular, it is well stocked with primitive idempotents, which are primitive idempotents also in A. It is not possible that P contains all these primitive idempotents; for then since it is closed it would contain e, the latter being a limit of sums of primitive idempotents. We pick a primitive idempotent g not in P. Let C be the closure of AgA. By Lemma

7, C is primitive. It is again right bounded,[2] and so Lemma 9 applies to show that C is discrete. That is, C is actually AgA and is a simple ring with minimal ideals. Since C is not in P, while both C and A/P are simple, we have a direct sum composition $A = C \oplus P$ at least in the algebraic sense. But this shows that C is isomorphic to A/P, hence is a finite simple ring with unit element; thus the direct sum is topological as well as algebraic. This completes the proof of Lemma 8.

Proof of Theorem 3. With Lemma 8 at hand, it is not difficult to analyze the structure of a locally compact right bounded semi-simple ring A. The basic tool is the compact open right ideal eA. We consider the closed primitive ideals in A; it follows from [5, Theorem 1] that their intersection is 0. Let J denote the intersection of those primitive ideals (necessarily open) that contain e. Since J contains eA, it is again open. We shall prove that J has the structure described in Theorem 3.

Let P_ι denote a typical closed primitive ideal not containing e. By Lemma 8 we have $A = h_\iota A \oplus P_\iota$, where $h_\iota A$ is a finite simple ring with unit element h_ι. We observe that h_ι lies in J. For let Q be a primitive ideal containing e. Then Q must contain either $h_\iota A$ or P_ι; but if it contains P_ι it must equal P_ι, since A/P_ι is simple. This is a contradiction, and so Q contains $h_\iota A$. Thus h_ι lies in J, which is the intersection of the Q's.

We define a mapping T of J into the complete direct sum of the rings $\{h_\iota A\}$ by sending $x \in J$ into $\{h_\iota x\}$. In the first place, T is faithful. For if $xT = 0$ then $h_\iota x = 0$, x lies in every P_ι; while $x \in J$ implies that x lies in every remaining primitive ideal. Thus $x = 0$. Next the range of T includes at least the algebraic direct sum of the rings $h_\iota A$, that is, the elements with all but a finite number of coordinates zero; this is an automatic consequence of the fact that $\{h_\iota A\}$ are distinct minimal two sided ideals in J. From the compactness of eA we then deduce that eA is (algebraically and topologically) the complete direct sum of the right ideals $h_\iota eA$. Incidentally $h_\iota e$ is non-zero, since otherwise e would lie in P_ι.

At this point we know that J contains the local direct sum of the rings $h_\iota A$ relative to the right ideals $h_\iota eA$. To complete the proof that J is precisely this local direct sum, only one thing more is needed; for any $y \in J$ we must show that all but a finite number of the elements $h_\iota y$ lie in $h_\iota eA$. Let U be a neighborhood of 0 (we can suppose that $U \subset eA$) such that $yU \subset eA$. Since eA is precisely the Cartesian product of $h_\iota eA$, it follows that

[2] Right boundedness is inherited by subrings.

U is unrestricted at all but a finite number of coordinates. That is, for all but a finite number of i we have $yh_ieA \subset h_ieA$, which implies $y \in h_ieA$. This completes the proof of Theorem 3.

The structure of J is thus completely determined, while A/J can be an arbitrary discrete semi-simple ring. We shall not attempt to study the ring extension problem that arises here, but it should at least be remarked that J need not be a direct summand of A. To get a counter-example, take J to be the local direct sum of finite simple rings relative to *proper* right ideals, and let A be the result of adjoining to J the unit element of the complete direct sum.

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CARTESIAN PRODUCTS OF REALS.*

By SAMUEL KAPLAN.

Introduction. Cartesian products of reals PR_λ have been finding increasing use in the theory of topological linear spaces, essentially because of their intimate connection with the weak topology of such spaces. An investigation of the basic properties of these products appears, therefore, to be desirable.

The primary problem is to free these spaces once and for all from their dependence on the special axes used to define them. In the present paper, this is attacked in two directions. The first direction, to which we devote part I, is that of finding intrinsic properties which characterize cartesian products of reals. We obtain two such characterizations. One of these is Lefschetz' *linear compactness* ([8], page 78), which he defined for his "linearly topologized vector spaces." Lefschetz showed (*cf.* [4]) that a linearly compact space of his type is a cartesian product of reals, the reals being taken as discrete. Clearly linear compactness as a concept can be defined equally well for ordinary topological linear spaces, and what we show is that the same structure theorem is obtained, viz.

A convex topological linear space is linearly compact if and only if it is a cartesian product of reals.

The second characterization is a not unexpected one:

A convex topological linear space is a cartesian product of reals if and only if it is complete in its weak topology.

The second direction, pursued in part III, is that of finding conditions, in a given cartesian product of reals PR_λ, for a set of one-dimensional linear subspaces $\{R_\alpha'\}$ to be an alternative set of axes for PR_λ. A necessary and sufficient condition is given in Theorem 11. One result in terms of matrices is the following (Corollary 2, Theorem 16):

Let $\{x^{(\alpha)}\}$ be a set of elements of PR_λ. Then a necessary condition that the $x^{(\alpha)}$'s generate an alternative set of axes is that the matrix of the coordinates of the $x^{(\alpha)}$'s be column-finite.

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In part III there are also given the answers to some immediate questions about closed linear subspaces of cartesian products of reals. For example, *every closed linear subspace has a complementary closed linear subspace,* or (Mackey) *the join of two closed linear subspaces is closed.*

Part II is devoted to studying the conjugate space of a cartesian product of reals and using it to establish the reflexivity of the latter. The topology of the conjugate space turns out to be simply the supremum of all convex topologies which can be imposed on it.

In closing we wish to point out that the present paper is heavily indebted to Mackey's thesis ([10], [11]).

I. Characterization Theorems.

1. Basic definitions and propositions. All our spaces are real linear spaces, and we will omit the word "real." We use the word "linear"—as applied to both spaces and functions—in its purely algebraic sense.

We denote the identity of a linear space X by θ. If A and B are any two subsets of X, $A+B$ consists of all x of the form $a+b$, where $a \varepsilon A$, $b \varepsilon B$. For any collection of subsets $\{A_\lambda\}$, ΣA_λ will denote the set of all finite sums of the form $a_{\lambda_1}+\cdots+a_{\lambda_k}$, where $a_{\lambda_i} \varepsilon A_{\lambda_i}$ $(i=1,\cdots,k)$.

In a topological linear space, any open set containing θ will be called a *nucleus.* As is customary, we will give the topology of a topological linear space by simply giving a nuclear base.

Given a collection $\{X_\lambda\}$ of topological linear spaces, the *cartesian product* $X=PX_\lambda$ of the X_λ's is the space of all collections $x=\{x_\lambda\}$, where

(i) for each λ, the coordinate x_λ of x is an element of X_λ;

(ii) multiplication by reals and addition are coordinatewise;

(iii) we take for a nuclear base all sets of the form $\{x \mid x_{\lambda_i} \varepsilon U_{\lambda_i};\ i=1,\cdots,k\}$, where $\lambda_1,\cdots,\lambda_k$ is any finite set of λ's, and U_λ is any nucleus in X_{λ_i} $(i=1,\cdots,k)$.

This topology is called the *Tychonoff,* or *product, topology.* We note that if the topology of every X_λ is convex, then that of X is also convex. For each λ_0, we identify X_{λ_0} with the subspace $\{x \mid x_\lambda=\theta$ for all $\lambda \neq \lambda_0\}$. As a result we have $X=(\Sigma X_\lambda)\dagger$ (the dagger means closure), any subproduct PX_μ $(\{\mu\} \subset \{\lambda\})$ can be written $PX_\mu=(\Sigma X_\mu)\dagger$, and a member of the nuclear base above can be written

$$U=\sum_{i=1}^{k} U_{\lambda_i}+(\sum_{\lambda \neq \lambda_i} X_\lambda)\dagger.$$

It is easily verified that if f is a continuous linear functional on PX_λ, then f is zero on all but a finite number of the X_λ's. It follows that every continuous linear functional on PX_λ can be written $f = f_{\lambda_1} + \cdots + f_{\lambda_k}$, where f_{λ_i} is a continuous linear function on X_{λ_i} $(i = 1, \cdots, k)$. And of course every such sum is a continuous linear functional on PX_λ.

If X is a topological linear space and X^* is the set of all continuous linear functionals on X, then the *weak topology* on X is defined as follows: For a nuclear base we take all sets of the form $\{x \mid |f_i(x)| < \epsilon\ (i = 1, \cdots, k)\}$, where $f_1, \cdots, f_k$ are any elements of X^*, and ϵ is any positive real number. It is easy to show that for $f_1, \cdots, f_k$ we need only consider linearly independent functionals.

If X is a linear space and F is the set of all linear functionals on X, then the *weak* topology* on F is defined as follows: For a nuclear base we take all sets of the form $\{f \mid |f(x_i)| < \epsilon\ (i = 1, \cdots, k)\}$, where $x_1, \cdots, x_k$ are any elements of X, and ϵ is any positive real number. Here also we need only consider linearly independent elements for $x_1, \cdots, x_k$.

We will need the following three (known) propositions.

(1) *Let X be a linear space, F the set of all linear functionals on X, and E any linear subspace of F which is total. Then under the weak* topology, $\bar{E} = F$.*

Proof. Consider any $f_0 \in F$. We have to show that given $\epsilon > 0$ and elements $x_1, \cdots, x_k$ of X, there exists an $f \in E$ such that $|f(x_i) - f_0(x_i)| < \epsilon$ for $i = 1, \cdots, k$. We will show in fact that there exists an $f \in E$ such that $f(x_i) = f_0(x_i)$ for $i = 1, \cdots, k$. If we let H be the subspace $\{f \in F \mid f(x_i) = 0\ (i = 1, \cdots, k)\}$, then our last statement says that $f_0 + H$ contains elements of E; it is this that we will prove. We assume that $x_1, \cdots, x_k$ are linearly independent; then they are linearly independent as linear functionals on F. It follows ([3], Théorème 1) that F/H is exactly k-dimensional. Now E is total; hence $x_1, \cdots, x_k$ are also linearly independent as linear functionals on E. Therefore $E/H \cap E$ is also exactly k-dimensional, and so is identical with F/H. It follows that $F = E + H$, which immediately gives the required conclusion.

(2) *The set of all linear functionals on a linear space X forms, under the weak* topology, a space topologically isomorphic to a cartesian product of reals PR_ξ, where for the index set $\{\xi\}$, we can take any Hamel basis in X.*

Proof. Let $\{\xi\}$ be any Hamel basis for X, that is, every $x \in X$ can be written uniquely in the form $x = t_1\xi_1 + \cdots + t_k\xi_k$ (the t_i's real numbers).

If f is a linear functional on X, f is completely determined by the values $f(\xi)$, ξ running through the Hamel basis. Denoting $f(\xi)$ by f_ξ, the mapping $f \to \{f_\xi\}$ is clearly an algebraic isomorphism of the set of all linear functionals onto PR_ξ. It is easy to see that the Tychonoff topology on PR_ξ is precisely the weak* topology.

(3) *Let $X = PR_\xi$ be a cartesian product of reals. Then no convex topology on X different from the Tychonoff has exactly the same set of continuous linear functionals.*

Proof. As we remarked earlier, each continuous linear functional on PR_λ (under the Tychonoff topology) has the form $f = f_{\lambda_1} + \cdots + f_{\lambda_k}$, where f_{λ_i} is a linear functional on R_{λ_i} $(i = 1, \cdots, k)$. This implies immediately that the resulting weak topology is precisely the Tychonoff topology. Now suppose we are given a convex topology on PR_λ which has exactly the same set of continuous linear functionals as the Tychonoff. From what we have just remarked, this topology must be at least as fine as the Tychonoff. We show it cannot be finer.

Let U be any symmetric convex nucleus in this topology. We will show that U contains all but a finite number of the R_λ's, whence it will follow easily that U contains a Tychonoff nucleus. Let $p(x)$ denote the pseudo-norm defined on PR_λ by U in the usual manner ([12], § 9). What we have to prove is that $p(x)$ is zero on all but a finite number of the R_λ's. Suppose not. Then there is a denumerable set $\{R_{\lambda_n}\}$ on which $p(x)$ is different from zero. By the Hahn-Banach theorem, for each R_{λ_n}, there is a linear function $f^{(n)}$ such that (i) $f^{(n)}$ has norm 1 relative to $p(x)$; (ii) $f^{(n)}$ attains its norm on R_{λ_n}. From (i), each $f^{(n)}$ is bounded relative to $p(x)$, hence continuous in the topology, and hence by assumption continuous in the Tychonoff topology. It follows that it is zero on all but a finite number of the R_λ's. As a result (by taking a subsequence if necessary) we can assume each $f^{(n)}$ equals zero on all R_{λ_m}, $m > n$. Now define $f = \sum_{n=1}^{\infty} (1/3^n) f^{(n)}$. f is bounded relative to $p(x)$ and hence, by the same argument as above, equals zero on all but a finite number of R_λ's. But a simple calculation, using (ii), shows that f differs from zero on all the R_{λ_n}'s. This gives a contradiction.

In Mackey's terminology (*cf.* [11]), the above theorem states that a cartesian product of reals is both relatively weak and relatively strong. The theorem could have been proved from general considerations of Mackey on ideals of pseudonorm sets. Katětov also has a proof ([6], Theorem (2.20)), but it seems to contain an error.

2. The characterization theorems. Given a linear space X, by a *linear variety* we mean any set of the form $x + X_0$, where X_0 is a linear subspace. A topological linear space is called *linearly compacct* if, given any collection of closed linear varieties with the finite intersection property, the entire collection has a non-empty intersection.

THEOREM 1. *Let X be a convex topological space. Then a necessary and sufficient condition that X be linearly compact is that it be topologically isomorphic to a cartesian product of reals.*

Proof. Assume X is linearly compact. Let X^* be the set of all continuous linear functionals on X, and $\tilde{X}$ the set of all linear functionals on X^*. By the usual identification, $X \subset \tilde{X}$.

We impose the weak* topology on $\tilde{X}$. Since the resulting topology on X (as a subspace of $\tilde{X}$) is its weak topology, it follows that a set closed in X under the topology of $\tilde{X}$ is closed in X under the original topology.

We show first that X is all of $\tilde{X}$. Consider any $\tilde{x} \varepsilon \tilde{X}$. Let $\{W_\alpha\}$ be the collection of linear subspace of $\tilde{X}$ defined as follows: for each α, W_α is the set of linear functionals on X^* which are zero on some finite numbers of points of X^*, and every such set appears as a W_α. The W_α's are each closed; therefore the linear varieties $\{\tilde{x} + W_\alpha\}$ are each closed, and their intersection is exactly $\tilde{x}$. From the proof of (1), the closed linear varieties of X $\{(\tilde{x} + W_\alpha) \cap X\}$ have the finite intersection property. Hence, since X is linearly compact, they have a non-empty intersection, which from the above must be $\tilde{x}$. Thus $\tilde{x} \varepsilon X$, and we have $X = \tilde{X}$.

Now from (2), X is topologically isomorphic to a cartesian product of reals. Also, X and $\tilde{X}$ have the same set of continuous linear functionals ([3], Théorème 2). It follows from (3) that the topology of X is identical with that of $\tilde{X}$.

The converse property, that a cartesian product of reals is linearly compact, is proved by the same type of argument as that used in the Tychonoff Theorem (*cf.* [8], Chap. II (27.2)).

THEOREM 2. *Let X be a convex topological linear space. Then a necessary and sufficient condition that X be topologically isomorphic to a cartesian product of reals is that X be complete under its weak topology.*

Proof. Assume X is complete under its weak topology, and let X^* and $\tilde{X}$ be the same as in the previous proof. Since the topology of X as a subset of $\tilde{X}$ is its weak topology, X is a complete subset of $\tilde{X}$ and therefore closed in $\tilde{X}$. But from (1), $X\dagger = \tilde{X}$, which gives us $X = \tilde{X}$. That the

topology of X is identical with that of X follows by the precise argument given in the previous proof.

The converse part of the theorem follows from the fact that a cartesian product of reals is complete, and its topology is its weak topology (*cf.* (3)).

We summarize the above results in one theorem (in connection with (c), *cf.* [11], Theorem 15).

THEOREM 3. *For a convex topological linear space X, the following four properties are equivalent:* (a) *X is linearly compact*; (b) *X is complete in its weak topology*; (c) *X is complete and its topology is identical with its weak topology*; (d) *X is topologically isomorphic to a cartesian product of reals.*

To disassociate the concept of cartesian product of reals from dependence on any special set of axes, we will call such a space an *entire topological linear space.*

II. Reflexivity Properties.

3. Weak products of reals. The *weak* or *combinatorial* (*cf.* [6]) *product* $Y = P^w Y_\lambda$ of a set of convex topological linear spaces is the subspace ΣY_λ of PY_λ (that is, the points with only a finite number of coordinates different from the identity), but topologized as follows: each collection $\{V_\lambda\}$, where V_λ is a convex nucleus of Y_λ, determines a convex set V in Y, viz. the convex envelope of $\bigcup V_\lambda$; the set of all V's obtained in this manner is taken as a nuclear base for the topology of Y.

Note that we have defined the weak product of convex spaces only. This is merely for convenience, since we consider only convex spaces in this paper. It is possible to define the weak product of any topological linear spaces.

What we are interested in is the weak product of reals, $Y = P^w S_\lambda$ (throughout the paper we will denote the images of reals which occur in cartesian products by R's and those which occur in weak products by S's). We will aproach this by means of an intrinsic definition. That is, we define a certain type of convex topological linear space, study a few of its properties, and then show that it is topologically isomorphic to a weak product of reals.

Let Y be a linear space. We take for a nuclear base the collection $\{V\}$ of *all* sets containing θ with the following two properties: 1°. V is convex; 2°. V intersects every one-dimensional linear subspace in an open interval. The resulting topology will be called the *totally fine* convex linear topology on Y.

That $\{V\}$ does satisfy the conditions for a nuclear base is easily shown. The nuclear base defining any convex topology in Y always has properties 1° and 2°; hence the totally fine topology is clearly the finest convex topology on Y, the supremum of all the convex topologies ([2], page 11). We will call a linear space under its totally fine topology a *totally fine convex topological linear space* or simply a *totally fine space.* As the following theorems show, the properties with which we will be principally concerned in such a space are determined entirely by its algebraic structure.

THEOREM 4. *If Y is a totally fine space, every linear functional on Y is continuous.*

Proof. The totally fine topology, being the supremum of all convex topologies on Y, is at least as fine as the weak topology defined by the set of all linear functionals of Y. (It is actually much finer.)

THEOREM 5. *In a totally fine space, every linear subspace is closed.*

From Theorem 4, every hyperplane is closed. Hence a linear subspace, being the intersection of all the hyperplanes containing it, is closed.

By a Hamel basis of a linear space Y, we (for the moment) mean, as usual, a set $\{\eta_\lambda\}$ such that every $y \in Y$ is a unique finite linear combination of η_λ's. For each λ, let S_λ be the one-dimensional linear subspace generated by η_λ. We will find it more convenient to work with the S_λ's than with the η_λ's; hence we will adopt the convention of calling the collection $\{S_\lambda\}$ also a Hamel basis. Explicitly, a collection of one dimensional linear subspaces $\{S_\lambda\}$ is a Hamel basis if $Y = \sum S_\lambda$ and, for every λ_0, $S_{\lambda_0} \cap \sum_{\lambda \neq \lambda_0} S_\lambda = \theta$.

Now let Y be a linear space and $\{S_\lambda\}$ a Hamel basis for Y. Every collection $\{I_\lambda\}$, where I_λ is an open interval of S_λ containing θ, determines a convex set V' in Y, viz. the convex envelope of $\bigcup I_\lambda$. Let us call any set V' obtained in this way an $\{S_\lambda\}$-convex-set. It is easy to show that the collection of all $\{S_\lambda\}$-convex-sets forms a nuclear base for a convex topology on Y.

THEOREM 6. *Let Y be a linear space and $\{S_\lambda\}$ a Hamel basis for Y. The topology defined by the collection of all $\{S_\lambda\}$-convex-sets is identical with the totally fine topology.*

It is enough to show that every set satisfying 1° and 2° contains an $\{S_\lambda\}$-convex-set. Suppose V satisfies 1° and 2°. For each λ, let $I_\lambda = V \cap S_\lambda$. Since V is convex, the convex envelope V' of $\bigcup I_\lambda$ lies in V.

Given any Hamel basis $\{S_\lambda\}$ in a totally fine space Y, we may also consider the weak product $P^w S_\lambda$. Clearly, under the identification of each S_λ

of the Hamel basis with the corresponding S_λ of $P^w S_\lambda$, we have an algebraic isomorphism of Y with $P^w S_\lambda$. Theorem 6 shows that this is also a topological isomorphism. Thus we have the

COROLLARY. *A totally fine space is topologically isomorphic to the weak product of any Hamel basis in it.*

Since the S_λ's defining a weak product of reals obviously form a Hamel basis for the weak product, it also follows from Theorem 6 that every weak product of reals is a totally fine space. We thus have

THEOREM 7. *For a convex topological linear space Y, the following two properties are equivalent:* (a) *Y is totally fine*; (b) *Y is topologically isomorphic to a weak product of reals.*

Also of interest is

THEOREM 8. *A totally fine space is complete.*

The proof is given in the Appendix (§ 10).

4. Reflexivity. In normed linear spaces, the conjugate space of a given space X is topologized essentially by means of the bounded sets of X. We do the same here.

Given a topological linear space X, we topologize the set X^* of all continuous linear functionals on X as follows. For every closed bounded set G of X, let

$$N(B) = \{f \in X^* \mid \sup_{x \in B} |f(x)| < 1\}. \tag{4}$$

(Note that $N(B)$ is automatically convex.) We take for a nuclear base in X^* the collection of all $N(B)$'s. X^* topologized in this fashion will be called the *conjugate* space of X, and will be denoted by the same symbol X^*.

We will need the following two lemmas.

LEMMA 1. *Let $X = PX_\lambda$ be a cartesian product of linear topological spaces. Then every bounded set B of X is contained in a bounded "cube," that is, a set of the form $(\Sigma B_\lambda)\dagger$, B_λ a bounded set in X_λ.*

Proof. For each λ_0, let $B_{\lambda_0} = \{x_{\lambda_0} \mid x \in B\}$. We show B_{λ_0} is bounded in X_{λ_0}. Consider any nucleus U_{λ_0} of X_{λ_0}. Since B is bounded, there exists a real number $t > 0$ such that the nucleus $t(U_{\lambda_0} + (\sum_{\lambda \neq \lambda_0} X_\lambda)\dagger)$ contains B. But then tU_{λ_0} contains B_{λ_0}, hence the latter is bounded. The set $(\Sigma B_\lambda)\dagger$ is then the required "cube."

LEMMA 2. *Let $Y = P^w Y_\lambda$ be a weak product of convex topological linear spaces. Then every bounded set B lies in a finite dimensional subspace $Y_{\lambda_1} + \cdots + Y_{\lambda_k}$.*

Proof. Suppose B lies in no such subspace. Then there exists a sequence $\{y^{(n)}\} \subset B$ and a sequence $\{\lambda_n\}$ such that $y_{\lambda_n}{}^{(n)} \neq \theta$ $(n = 1, 2, \cdots)$. For each λ_n, choose a convex nucleus V_{λ_n} such that $y_{\lambda_n}{}^{(n)} \notin nV_{\lambda_n}$; and for all other λ's, take $V_\lambda = Y_\lambda$. Then if V is the convex envelope of $\bigcup V_\lambda$, $B \not\subset tV$ for any $t > 0$.

In what follows, if y is a linear functional on a linear space X, we will sometimes use the notation xy or yx to denote $y(x)$.

We now proceed with the proof of reflexity.

LEMMA 3. *Let $\{X_\lambda\}$ be a set of topological linear spaces. Then*

$$(PX_\lambda)^* = P^w X_\lambda{}^*.$$

Proof. For each λ, let us write $X_\lambda{}^* = Y_\lambda$. That $P^w Y_\lambda$ is the set of continuous linear functionals on PX_λ, under the definition $xy = \Sigma\, x_\lambda y_\lambda$, is proved in ([6], Theorem (2. 2)). (It also follows from a remark we made in § 1). To establish the lemma, it is enough, from Lemma 1, to show that if B is of the form $B = (\Sigma\, B_\lambda)\dagger$ in PX_λ, then $N(B)$ is exactly the convex envelope of $\bigcup V_\lambda$, where $V_\lambda = N(B_\lambda)$ in Y_λ. This is readily verified by calculation.

LEMMA 4. *Let $\{Y_\lambda\}$ be a set of convex topological linear spaces. Then $(P^w Y_\lambda)^* = PY_\lambda{}^*$.*

Proof. For each λ, let us write $Y_\lambda{}^* = X_\lambda$. That PX_λ is the set of continuous linear functionals on $P^w Y_\lambda$ (with $xy = \Sigma\, x_\lambda y_\lambda$) is easily proved (*cf.* [6], Theorem 2. 2). To establish the lemma it is enough, from Lemma 2, to show that if B is a bounded set in $Y_{\lambda_1} + \cdots + Y_{\lambda_k}$, then $N(B)$ in PX_λ is exactly the set $U(\lambda_1, \cdots, \lambda_k) + (\sum_{\lambda \neq \lambda_i} X_\lambda)\dagger$, where $U(\lambda_1, \cdots, \lambda_k) = N(B)$ in $X_{\lambda_1} + \cdots + X_{\lambda_k}$. Again this is readily verified by calculation.

If a topological linear space X has the property that $(X^*)^* = X$ (not only set-theoretically, but topologically) we will say that X is *reflexive.* (In this case X is convex). From Lemmas 3 and 4 we have immediately

LEMMA 5. *If $X = PX_\lambda$ and, for each λ, X_λ is reflexive, then X is also. If $Y = P^w Y_\lambda$ and, for each λ, Y_λ is reflexive, then Y is also.*

From this we conclude

THEOREM 9. *An entire topological linear space (a totally fine space) is reflexive, and its conjugate space is a totally fine space (an entire topological linear space).*

COROLLARY. *The converse of* (2) *holds, that is, an entire topological linear space is the set of all linear functionals on a linear space topologized by the weak* topology.*

The concept of conjugacy and reflexiveness can be formulated with closed bounded sets replaced by compact sets in (4). Lemmas 1 through 5 and Theorem 9 are easily seen to hold in this case too. Indeed we have a stronger conclusion. In the reals, the compact sets are identical with the closed bounded sets. It is easy to show, using Lemmas 1 and 2 that the same is true for cartesian and weak products of reals, that is

LEMMA 6. *In entire topological linear spaces and in totally fine spaces, the compact sets are identical with the closed bounded sets.*

III. Transformations of Axes.

5. Sets of axes. Let X be a linear space. Two linear subspaces X_1, X_2 are called *complementary* if $X_1 + X_2 = X$ and $X_1 \cap X_2 = \theta$. In such case it is trivial that every $x \in X$ can be written $x = x_1 + x_2$, $x_1 \in X_1$, $x_2 \in X_2$ in only one way.

Definition. Let X be a topological linear space. A set $\{R_\lambda\}$ of one-dimensional linear subspaces of X will be called *a set of axes for* X if properties (I) and (II) below hold.

(I) *For each partition of the index set* $\{\lambda\}$ *into two disjoint sets,* $\{\lambda\} = \{\mu\} \cup \{\nu\}$, *the closed linear subspaces* $(\Sigma R_\mu)\dagger$ *and* $(\Sigma R_\nu)\dagger$ *are complementary.*

Consider any $x \in X$. For every λ_0, R_{λ_0} and $(\sum_{\lambda \neq \lambda_0} R_\lambda)\dagger$ are complementary, by (I); hence x can be written uniquely $x = x_{\lambda_0} + x'$, $x_{\lambda_0} \in R_{\lambda_0}$, $x' \in (\sum_{\lambda \neq \lambda_0} R_\lambda)\dagger$. We will call x_{λ_0} the λ_0-*component* of x (with respect to the set $\{R_\lambda\}$). Now the finite subsets of the index set $\{\lambda\}$ are directed by inclusion, hence the finite sums $x_{\lambda_1} + \cdots + x_{\lambda_k}$ of the components of x form a directed set, and we can consider the question of the (Moore-Smith) convergence of this directed set. We can now state property (II).

(II) *For every $x \in X$, the directed set of all finite sums of the components of x converges to x. We denote this by $x = \Sigma x_\lambda$.*

We establish some immediate properties. In the following, $\{R_\lambda\}$ is a fixed set of axes.

(5) *For any two elements x, x' of X, $x = x'$ if and only if $x_\lambda = x'_\lambda$ for all λ.*

The necessity follows from the uniqueness of components; the sufficiency from the uniqueness of convergence.

(6) *If $\{\mu\}$ is a subset of $\{\lambda\}$, then a necessary and sufficient condition that $x \in (\Sigma R_\mu)\dagger$ is that $x_\lambda = \theta$ for all λ not in $\{\mu\}$.*

Proof. Assume $x \in (\Sigma R_\mu)\dagger$, and consider any λ_0 not in $\{\mu\}$. We can write $x = x + \theta$, where $x \in (\Sigma R_\mu)\dagger \subset (\sum_{\lambda \neq \lambda_0} R_\lambda)\dagger$ and $\theta \in R_{\lambda_0}$; hence from the uniqueness of components, $x_{\lambda_0} = \theta$. Conversely, suppose $x_\lambda = \theta$ for all λ not in $\{\mu\}$. Then $x = \Sigma x_\lambda = \Sigma x_\mu \in (\sum R_\mu)\dagger$.

(7) *Let $\{\mu\}$ be a subset of $\{\lambda\}$ and $x \in X$. We have $x = y + z$, where $y \in (\Sigma R_\mu)\dagger$ and $z \in (\sum_{\lambda \neq \mu} R_\lambda)\dagger$. Then* (i) *$y_\mu = x_\mu$ for all μ;* (ii) *$y_\lambda = \theta$ for all λ not in $\{\mu\}$.*

(ii) was proved in (6). We prove (i). For any μ_0, $y = y_{\mu_0} + y'$, $y' \in (\sum_{\lambda \neq \mu_0} R_\lambda)\dagger$. Hence $x = y_{\mu^0} + (y' + z)$, $y' + z \in (\sum_{\lambda \neq \mu_0} R_\lambda)\dagger$. It follows that $x_{\mu_0} = y_{\mu_0}$. (Note that the proof of (7) uses only property (I)).

(8) *Suppose that for each σ, Λ_σ is a subset of $\{\lambda\}$. If $\{\mu\} = \bigcap \Lambda_\sigma$, then $(\Sigma R_\mu)\dagger = \bigcap_\sigma (\sum_{\lambda \in \Lambda\sigma} R_\lambda)\dagger$* (If $\bigcap \Lambda_\sigma$ is empty, we write $(\Sigma R_\mu)\dagger = \theta$).

Proof. The left side is clearly contained in the right. To show the inverse inclusion, consider $x \notin (\Sigma R_\mu)\dagger$. From (6), $x_{\lambda_0} \neq \theta$ for some λ_0 not in $\{\mu\}$. Choose σ such that $\lambda_0 \notin \Lambda_\sigma$. Then, from (6) again, $x \notin (\sum_{\lambda \in \Lambda\sigma} R_\lambda)\dagger$, hence, a fortiori, x is not an element of the right side.

THEOREM 10. *In a totally fine space Y,* (a) *every Hamel basis is a set of axes;* (b) *every set of axes is a Hamel basis;* (c) *property* (I) *implies property* (II).

Proof. Let $\{S_\lambda\}$ be a Hamel basis. Then, for every partition of $\{\lambda\}$ into two disjoint sets, $\{\lambda\} = \{\mu\} \cup \{\nu\}$, ΣS_μ and ΣS_ν are complementary. But from Theorem 5, $\Sigma S_\mu = (\Sigma S_\mu)\dagger$ and $\Sigma S_\nu = (\Sigma S_\nu)\dagger$; therefore property

(I) holds. Since every $y \in Y$ lies in a finite sum of the S_λ's, property (II) is trivially satisfied. This proves (a).

Now suppose $\{S_\lambda\}$ is a set of axes. This means $Y = (\Sigma S_\lambda)\dagger = \Sigma S_\lambda$ and for each λ_0, $S_{\lambda_0} \cap (\sum_{\lambda \neq \lambda_0} S_\lambda)\dagger = \theta$; thus $\{S_\lambda\}$ is a Hamel basis. This proves (b). In proving (b), we have used only property (I), hence, applying (a), we have (c).

If X is a topological linear space and X_0 a linear subspace, then by $X_0{}^\perp$ we will mean the set of all continuous linear functionals which are zero on X_0. The following proposition is well known.

(9) *Let X and Y be convex topological linear spaces such that each is the set of continuous linear functionals on the other. Then*

(a) *if X_0 is a linear subspace of X, $(X_0{}^\perp)^\perp = \bar{X}_0$;*

(b) *if $\{X_\lambda\}$ is a collection of linear subspaces of X, then $(\Sigma X_\lambda)^\perp = \cap X_\lambda{}^\perp$;*

(c) *if $(X_1 + X_2)\dagger = X$, then $X_1{}^\perp \cap X_2{}^\perp = \theta$;*

(d) *if $\bar{X}_1 \cap \bar{X}_2 = \theta$, then $(X_1{}^\perp + X_2{}^\perp)\dagger = Y$.*

Suppose X and Y are convex topological linear spaces such that each is the set of continuous linear functionals of the other. Let $\{R_\lambda\}$ be a set of axes in X. Then a set of axes $\{S_\lambda\}$ in Y (with the same index system) will be said to be *orthogonal* to $\{R_\lambda\}$ if, for every λ_0,

$$S_{\lambda_0} = (\sum_{\lambda \neq \lambda_0} R_\lambda)^\perp \tag{10}$$

Note that there cannot be more than one set of axes in Y orthogonal to $\{R_\lambda\}$, since (10) deterimnes each S_{λ_0} uniquely.

(11) *Under the above conditions*:

(i) *For every partition of $\{\lambda\}$ into two disjoint sets,*

$$\{\lambda\} = \{\mu\} \cup \{\nu\}, \qquad (\Sigma S_\nu)\dagger = (\Sigma R_\mu)^\perp;$$

(ii) *$\{R_\lambda\}$ is in turn orthogonal to $\{S_\lambda\}$.*

Proof. From (9a), for each λ_0, $(\sum_{\lambda \neq \lambda_0} R_\lambda)\dagger = S_{\lambda_0}{}^\perp$. Hence, from (9b) and (8), $(\Sigma R_\mu)\dagger = (\Sigma S_\nu)^\perp$. Applying (9a) again gives (i). To establish (ii) we put $\{\mu\} = \lambda_0$ in (i) and obtain $(\sum_{\lambda \neq \lambda_0} S_\lambda)\dagger = R_{\lambda_0}{}^\perp$. Then another application of (9a) gives $R_{\lambda_0} = (\sum_{\lambda \neq \lambda_0} S_\lambda)\dagger^\perp$, which is (ii).

6. Axes in entire topological linear spaces. It is of course trivial that

LEMMA 7. *In a cartesian product of reals PR_λ, the collection $\{R_\lambda\}$ is a set of axes and is orthogonal to the collection $\{S_\lambda\}$ of the conjugate space $P^w S_\lambda$.*

The question arises: What other sets of axes are possible for PR_λ besides the set of R_λ's defining it? One answer is given by

THEOREM 11. *Let X be an entire topological linear space and Y its conjugate totally fine space. Then there is a one-one correspondence between the sets of axes in X and the Hamel bases in Y, the correspondence being given by orthogonality.*

Proof. Consider any Hamel basis $\{S_\lambda\}$ in Y. For each λ_0, let $R'_{\lambda_0} = (\sum_{\lambda \neq \lambda_0} S_\lambda)^\perp$. We show that the resulting collection $\{R'_\lambda\}$ is a set of axes in X. Now Y is topologically isomorphic to $P^w S_\lambda$ (Corollary to Theorem 6), and therefore its conjugate space is PR_λ. Hence, since X is conjugate to Y, there is determined a topological isomorphism of X on PR_λ. Moreover, since $(\sum_{\lambda \neq \lambda_0} S_\lambda)^\perp$ is R'_{λ_0} in X and R_{λ_0} in PR_λ, we must have $R'_{\lambda_0} \to R_{\lambda_0}$ under this isomorphism. It follows that $\{R'_\lambda\}$ is a set of axes in X.

Conversely, consider any set of axes $\{R_\lambda\}$ in X. For each λ_0, let $S_{\lambda_0} = (\sum_{\lambda \neq \lambda_0} R_\lambda)^\perp$. We show that $\{S_\lambda\}$ is a Hamel basis in Y. In the proof of (11) we did not use the fact that $\{S_\lambda\}$ was a set of axes, hence we can apply the results here. This gives first (using (9a)) that for each λ_0, $S_{\lambda_0} = (\sum_{\lambda \neq \lambda_0} R_\lambda)\dagger^\perp$, $\sum_{\lambda \neq \lambda_0} S_\lambda = R_{\lambda_0}{}^\perp$, hence (using (9c)) that $S_{\lambda_0} \cap \sum_{\lambda \neq \lambda_0} S_\lambda = \theta$. Secondly, it gives $\sum_\lambda S_\lambda = \theta^\perp = Y$, and thus $\{S_\lambda\}$ is a Hamel basis.

That the correspondence between the sets of axes in X and the Hamel bases in Y is one-one follows from the uniqueness of orthogonality. This completes the proof.

COROLLARY. *If $\{R_\lambda\}$ and $\{R'_\alpha\}$ are two sets of axes for an entire topological linear space, then the index sets $\{\lambda\}$ and $\{\alpha\}$ have the same cardinality.*

This follows from the fact that all Hamel bases in Y have the same cardinality ([8], Chap. II (24.1)).

From the proof of Theorem 11, we also have

THEOREM 12. *If X is an entire topological linear space and $\{R_\lambda\}$ a collection of one-dimensional linear subspaces, then X is topologically isomorphic to PR_λ with $R_\lambda \to R_\lambda$ if and only if $\{R_\lambda\}$ is a set of axes.*

COROLLARY. *If X is an entire topological linear space and $\{R_\lambda\}$ is a set of axes, then the sets of the form*

$$U(\lambda_1, \cdots, \lambda_k) + (\sum_{\lambda \neq \lambda_i} R_\lambda)\dagger, \tag{12}$$

where $U(\lambda_1, \cdots, \lambda_k)$ is a nucleus in the finite dimensional subspace $\sum_{i=1}^{k} R_{\lambda_i}$, form a nuclear base for X.

7. Subspaces of entire topological linear spaces.

THEOREM 13. *Let X be an entire topological linear space. If X_0 is a closed subspace, there exists a set of axes $\{R_\lambda\}$ such that $X_0 = (\Sigma R_\mu)\dagger$, where $\{\mu\}$ is a subset of $\{\lambda\}$.*

Proof. Let Y be the totally fine space conjugate to X. Choose a Hamel basis $\{S_\lambda\}$ for Y such that $X_0^\perp = \Sigma S_\nu$, where $\{\nu\}$ is a subset of $\{\lambda\}$. Denote the remaining λ's by $\{\mu\}$. Now let $\{R_\lambda\}$ be the set of axes in X orthogonal to $\{S_\lambda\}$. Then from (11), $(\Sigma R_\mu)\dagger = (\Sigma S_\nu)^\perp = X_0$.

If X is a topological linear space and X_0 is a linear subspace, a *projection* of X on X_0 is a continuous linear mapping E of X onto X_0 such that $E^2(X) = E(X)$ for all x. In such case, as is easily shown, $E^{-1}(\theta)$ is complementary to X_0. In a cartesian product PX_λ it is obvious that the natural mapping onto any subproduct PX_μ is a projection. Now in Theorem 13, X is topologically isomorphic to PR_λ, which gives us

THEOREM 14. *In an entire topological linear space, if X_0 is a closed linear subspace, there exists a projection on X_0 (and hence X_0 has a complementary closed linear subspace).*

It is also easily shown that

THEOREM 15. *If X is an entire topological linear space, and X_1, X_2 are complementary closed linear subspaces, then there exists a set of axes $\{R_\lambda\}$ such that $X_1 = (\Sigma R_\mu)\dagger$, $X_2 = (\Sigma R_\nu)\dagger$, where $\{\lambda\} = \{\mu\} \cup \{\nu\}$.*

In any topological linear space X, if X/X_0 is finite dimensional, the subspace X_0 is said to have *finite deficiency* ([10], Chap. I, § 1).

THEOREM 16. *Let X be an entire topological linear space and $\{R_\lambda\}$ a set of axes. Then if X_0 is any closed linear subspace of finite deficiency, X_0 contains all but a finite number of the R_λ's.*

Proof. Let $\{S_\lambda\}$ be the Hamel basis in the conjugate space Y which

is orthogonal to $\{R_\lambda\}$. $X_0^\perp$ is finite dimensional ([3], Théorème 1); hence there exixst $\lambda_1, \cdots, \lambda_k$ such that $X_0^\perp \subset \sum_{i=1}^{k} S_{\lambda_i}$. Since $(\sum_{\lambda \neq \lambda_i} R_\lambda)\dagger = (\sum_{i=1}^{k} S_{\lambda_i})^\perp$, it follows that $(\sum_{\lambda \neq \lambda_i} R_\lambda)\dagger \subset (X_0)^\perp)^\perp = X_0$.

COROLLARY 1. *If $\{R_\lambda\}$, $\{R'_\alpha\}$ are two sets of axes for an entire topological linear space, then for any finite set $\lambda_1, \cdots, \lambda_k$, $(\sum_{\lambda \neq \lambda_i} R_\lambda)\dagger$ contains all but a finite number of the R'_α's.*

In terms of matrices, this can be stated as follows.

COROLLARY 2. *Given PR_λ, suppose $\{z^{(\alpha)}\}$ generates a set of axes (that is, the set of one-dimensional linear subspaces determined by the respective $z^{(\alpha)}$'s is a set of exes). Then the matrix of the coordinates of the $z^{(\alpha)}$'s is column-finite.*

Given two closed linear subspaces X_1, X_2 of a topological linear space two questions that arise are the following: "Is $X_1 + X_2$ closed?" and "If $X_1 \cap X_2 = \theta$, is $X_1 + X_2$ topologically isomorphic to the cartesian product of X_1 and X_2?" In general, the answer is no to both questions ([10], pp. 173-174). However Mackey has shown that in an entire topological linear space, the answer to both questions is yes. Because of its interest we state this as a theorem.

THEOREM 17. (*Mackey*). *In an entire topological linear space, if X_1, X_2 is any pair of closed linear subspaces, then* (a) *$X_1 + X_2$ is closed*; (b) *if $X_1 \cap X_2 = \theta$, then $X_1 + X_2$ is topologically isomorphic to the cartesian product of X_1 and X_2.*

The theorem follows from ([10], Corollary 3 of Theorem III-6, Theorem III-7, and the paragraph preceding Theorem III-6).

THEOREM 18. *If the index set $\{\lambda\}$ has cardinality $2^{\aleph_0}$, then $X = PR_\lambda$ contains a linear subspace of dimension $\aleph_0$ which is dense in X (thus X is separable).*

Proof. Since all cartesian products of the same number of images of reals are topologically isomorphic, it is enough to exhibit one example with the above property. Let $Z = P^w R'_n$ $(n = 1, 2, \cdots)$ considered simply algebraically. Let Y be the space of all linear functionals on Z. The cardinality of any Hamel basis in Y is clearly $2^{\aleph_0}$. Now let X be the space of all linear functionals on Y, with the weak* topology. From (2), X has the form PR_λ, where the cardinality of $\{\lambda\}$ is $2^{\aleph_0}$; and from (1), Z is dense in X.

It is obvious the above theorem holds with $\aleph_0$ replaced by any infinite cardinal.

Appendix.

8. Sets of axes in Banach spaces. An interesting question on sets of axes is the following: does property (II) follow from property (I)? We show here that such is the case in Banach spaces. In fact we have a stronger result:

THEOREM 19. *Let X be a Banach space and $\{R_\lambda\}$ a set of one-dimensional linear subspaces having property* (I). *Then for each $x \in X$, all but a countable number $\{x_n\}$ of the components are zero and* $\lim_{n\to\infty} \| x - \sum_{i=1}^{n} x_i \| = 0$.

To prove this we make use of a theorem proved independently by E. R. Lorch ([9], Theorems (2.1) and (2.2)) and H. Kober [7]. This theorem says essentially that in a Banach space, properties (a) and (b) of Theorem 17 are equivalent. However we state it in a form adapted to the present requirements.

(13) (*Lorch-Kober*) *Let X be a Banach space and X_1, X_2 two complementary closed subspaces. For every $x \in X$, let $x = x_1 + x_2$, $x_1 \in X_1$, $x_2 \in X_2$, be the unique representation of x with respect to X_1, X_2. Then the mapping of X onto X_1 given by $x \to x_1$ is continuous.*

We proceed with the proof of Theorem 19. Consider $x \in X$. For any subset $\{\mu\}$ of the index set $\{\lambda\}$, we have $x = x' + x''$, where $x' \in (\sum R_\mu)\dagger$ and $x'' \in (\sum_{\lambda \notin \{\mu\}} R_\lambda)\dagger$. We will call x' the *component* of x in $(\sum R_\mu)\dagger$. Now $x \in (\sum R_\lambda)\dagger$; therefore there exists $y^{(1)} \in \sum_{n=1}^{n_1} R_{\lambda_n}$ such that $\| x - y^{(1)} \| < 1$. Denote $\sum_{n=1}^{n_1} R_{\lambda_n}$ by Y_1 and the component of x in Y_1 by $x^{(1)}$. It follows that

$$(x - x^{(1)}) \in (\sum_{\lambda \neq \lambda_i} R_\lambda)\dagger;$$

hence there exist $y^{(2)} \in \sum_{n=n_1+1}^{n_2} R_{\lambda_n}$ such that $\| (x - x^{(1)}) - y^{(2)} \| < 1/2$. Denote $\sum_{n=n_1+1}^{n_2} R_{\lambda_n}$ by Y_2 and the component of x in Y_2 by $x^{(2)}$. Continuing in this fashion, we obtain sequences $\{Y_m\}$, $\{y^{(m)}\}$, $\{x^{(m)}\}$ such that

(i) $x^{(m)}$ is the component of x in Y_m;

(ii) $\lim_{M\to\infty} \| (x - \sum_{m=1}^{M-1} x^{(m)}) - y^{(M)} \| = 0$.

We will show that $\lim_{M\to\infty} \| x - \sum_{m=1}^{M} x^{(m)} \| = 0$, which will prove the theorem (Cf. (6)). Now $\| x - \sum_{m=1}^{M} x^{(m)} \| \leqq \| (x - \sum_{m=1}^{M-1} x^{(m)}) - y^{(M)} \| + \| y^{(M)} - x^{(M)} \|$; hence from (ii), it is enough to show that $\lim_{M\to\infty} \| y^{(M)} - x^{(M)} \| = 0$.

Suppose this last is not true. Then there exists $\epsilon > 0$ and a subsequence $\{M_p\}$ such that

(iii) $$\| y^{(M_p)} - x^{(M_p)} \| \geqq \epsilon \text{ for } p = 1, 2, \cdots,$$

and by dropping out terms if necessary, we can assume:

(iv) No two successive M's are in the subsequence $\{M_p\}$.

Now from (ii), $\lim_{M\to\infty} \| (\sum_{m=1}^{M-1} x^{(m)} + y^{(M)}) - (\sum_{m=1}^{M} x^{(m)} - y^{(M+1)}) \| = 0$, and this reduces to

(v) $$\lim_{M\to\infty} \| y^{(M)} - x^{(M)} - y^{(M+1)} \| = 0.$$

Let $Z = (\Sigma X_{M_p})\dagger$. From (iv), the component of $y^{(M_p)} - x^{(M_p)} - y^{(M_p+1)}$ in Z is $y^{(M_p)} - x^{(M_p)}$. Hence from (v) and the Lorch-Kober Theorem above, $\lim_{p\to\infty} \| y^{(M_p)} - x^{(M_p)} \| = 0$, which contradicts (iii).

9. In contrast to the above, we now show that in an entire topological linear space, property (I) does *not* imply property (II). Let Z, Y, X be the spaces in the proof of Theorem 18. We show first that the set $\{R'_n\}$ has property (I) in X. Let $\{n\} = \{p\} \cup \{q\}$ be a partition of $\{n\}$. That $(\Sigma R'_p)\dagger + (\Sigma R'_q)\dagger = X$ follows from Theorem 17 and Theorem 18. We show that $(\Sigma R'_p)\dagger \cap (\Sigma R'_q)\dagger = \theta$ by showing that $(\Sigma R'_p)^{\perp} + (\Sigma R'_q)^{\perp} = Y$. Now Y can be considered as the space of all sequences, $y = (y_1, y_2, \cdots, y_n, \cdots)$, and clearly $(\Sigma R'_p)^{\perp}$ consists of exactly the sequences with zero in the p-positions while $(\Sigma R'_q)^{\perp}$ consists of exactly the sequences with zero in the q-positions. It is obvious that these two classes of sequences generate Y.

Thus the set R'_n has property (I). If it had property (II), it would constitute a set of axes, which would contradict the Corollary to Theorem 11. This completes the proof.

10. Completeness of totally fine spaces. We present here the proof of Theorem 8. From Theorem 7, it is enough to give the proof for a weak product of reals $Y = P^w S_\lambda$. By definition an element V of the nuclear base is the convex envelope of a set $\bigcup_\lambda I_\lambda$, where I_λ is an open interval of S_λ

containing zero. Clearly, we can confine ourselves to I_λ's which are symmetric around zero, and we shall do so. If we denote the half-length of each I_λ defining V by e_λ (e_λ may possibly be ∞), then V is completely determined by the e_λ's as follows:

$$V = \{y \mid \Sigma \mid y_\lambda|/e_\lambda < 1\}. \tag{14}$$

(Since only a finite number of the y_λ's in any one y are different from zero, the above sum is always well defined.)

Now consider a Cauchy system $\{y^{(\alpha)}\}$ in Y. This means that given any collection $\{e_\lambda\}$, there exists α such that

(i) $\qquad \beta > \alpha$ implies $\Sigma \mid y_\lambda{}^{(\beta)} - y_\lambda{}^{(\alpha)}|/e_\lambda < 1.$

This gives, in particular, that

(ii) $\qquad \beta > \alpha$ implies $\mid y_\lambda{}^{(\beta)} - y_\lambda{}^{(\alpha)} \mid < e_\lambda$ for every λ.

This latter implies that the coordinates of the $y^{(\alpha)}$'s converge on each S_λ. We denote the limit on each S_λ by y_λ.

We show first that $y_\lambda = 0$ for all but a finite number of λ's. Suppose not. Then for some denumerable set of λ's, $\{\lambda_n\}$, the y_{λ_n}'s are all different from zero. Choose $e_{\lambda_n} = \frac{1}{2} \mid y_{\lambda_n} \mid$ $(n = 1, 2, \cdots)$ and for the rest of the λ's, choose $e_\lambda = \infty$. Let α satisfy (ii) with respect to the set $\{e_\lambda\}$. Then since $y_\lambda{}^{(\alpha)} = 0$ for all but a finite number of λ's, there is an n such that $y_{\lambda_n}{}^{(\alpha)} = 0$. But then, for this n, $\beta > \alpha$ implies $y_{\lambda_n}{}^{(\beta)} < \frac{1}{2} \mid y_{\lambda_n} \mid$, which contradicts the fact that $\lim_\alpha y_{\lambda_n}{}^{(\alpha)} = y_{\lambda_n}$.

Since only a finite number of the y_λ's are thus different from zero, $y = \sum_\lambda y_\lambda$ is an element of Y. We complete the proof by showing that $\lim y^{(\alpha)} = y$. Given any set $\{e_\lambda\}$, choose α to satisfy (i). Let $\lambda_1, \cdots, \lambda_k$ be the set of indices for which the coordinates of either y or $y^{(\alpha)}$ are different from zero. Since $\sum_{i=1}^{k} S_{\lambda_i}$ is finite dimensional, we can find $\alpha' > \alpha$ such that

(iii) $\qquad \beta > \alpha'$ implies $\sum_{i=1}^{k} \mid y_{\lambda_i}{}^{(\beta)} - y_{\lambda i} \mid /e_{\lambda i} < 1.$

For the rest of the λ's we have (since $y_\lambda = 0 = y_\lambda{}^{(\alpha)}$)

(iv) $\qquad \beta > \alpha'$ implies $\sum_{\lambda \neq \lambda_i} \mid y_\lambda{}^{(\beta)} - y_\lambda \mid /e_\lambda = \sum_{\lambda \neq \lambda_i} \mid y_\lambda{}^{(\beta)} \mid /e_\lambda$

$$= \sum_{\lambda \neq \lambda_i} \mid y_\lambda{}^{(\beta)} - y_\lambda{}^{(\alpha)} \mid /e_\lambda \leqq \sum_{\text{all } \lambda} \mid y_\lambda{}^{(\beta)} - y_\lambda{}^{(\alpha)} \mid /e_\lambda < 1,$$

from (i). Combining (iii) and (iv), we obtain that

$$\beta > \alpha' \text{ implies } \sum_{\lambda} | y_\lambda^{(\beta)} - y_\lambda | / e_\lambda < 2.$$

This completes the proof.

Remark. It may be of interest to note that a weak product of a countably infinite set of reals $P^w S_n$ furnishes a simple example of a complete topological linear space which is of the first category.

As a final remark, we point out that the completeness of entire and totally topological linear spaces are, as is not hard to see, just special cases of the following.

If $\{X_\lambda\}$ is a set of complete topological linear spaces, then PX_λ and $P^w X_\lambda$ are also complete.

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NORMAL CURVATURE OF A VECTOR FIELD.*

By T. K. PAN.

1. Introduction. Let v be a vector field in a surface in an ordinary space. The derived vector of v along a curve C on the surface is called *the absolute curvature vector of the field with respect to the curve.* This derived vector can be decomposed into a component tangential and a component normal to the surface. The former, called *the associate curvature vector of v along C* or *the angular spread vector of v along C,* was studied by W. C. Graustein [1] and was later generalized to Riemannian space by R. M. Peters [2]. This paper deals with the latter component, which is called *the normal curvature vector of the vector field v with respect to the curve C.* The following terms are defined: normal curvature of a vector field with respect to a curve, asymptotic direction of a vector field, asymptotic line of a vector field, principal direction of a vector field, line of curvature of a vector field, principal curvature of a vector field, indicatrix of a vector field, curve of a vector field, and principal vector field of the surface; new characterizations of the principal curvatures of the surface and the lines of curvature of the surface are derived; and some properties analogous to those connected with normal curvature of a curve on the surface are studied, first for vector fields in a surface in an ordinary space, and then extended to fields in V_n in V_{n+1} and to fields in V_n in V_m.

The notation of Eisenhart [3], [4] will be used for the most part except that $\Gamma^{\alpha}_{\beta\gamma}$ will be employed for Christoffel symbols of the second kind.

2. Definitions. Let $S: x^i = x^i(u^1, u^2)$, $i = 1, 2, 3$ be a real proper analytic surface in an ordinary space with reference to a rectangular cartesian coordinate system. Associate to each point of S an arbitrary but fixed unit vector v^i such that $v^i = p^{\alpha} x^i_{,\alpha}$, $g_{\alpha\beta} p^{\alpha} p^{\beta} = 1$, and v^i are real analytic functions of u^{α}. It will be denoted throughout by v or p at appropriate places. Let $C: u^{\alpha} = u^{\alpha}(s)$, $\alpha = 1, 2$ be a curve on S. Then to each point $P(x^i)$ of C there is associated a unit vector v. Consequently, v^i are functions of the arc length s along the curve C. The derived vector of v along C at P is defined by

$$(2.1) \qquad dv^i/ds = {}_v k = {}_v\kappa w^i = x^i_{,\alpha} p^{\alpha}_{,\gamma} du^{\gamma}/ds + (p^{\delta} d_{\delta\gamma} du^{\gamma}/ds) X^i,$$

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where $p^{\alpha}_{;\gamma}$ is the covariant derivative of the contravariant vector p^{α} based upon $g_{\alpha\beta}$, where X^{i} are the components of the unit normal to S at P, and where ${}_{v}\kappa$, called *the absolute curvature of the vector field v with respect to C* at P, is the magnitude of the derived vector ${}_{v}k$ in whose sense the unit vector w^{i} is taken. Let ω be the angle between w^{i} and X^{i}. We have from (2.1) ${}_{v}\kappa\cos\omega = p^{\delta}d_{\delta\gamma}du^{\gamma}/ds$. If ${}_{v}\kappa$ does not vanish, the right side of the above equation is positive or negative according as $0 \leqq \omega < \frac{1}{2}\pi$ or $\pi \geqq \omega > \frac{1}{2}\pi$. Let ${}_{v}\kappa_{n} = ep^{\delta}d_{\delta\gamma}du^{\gamma}/ds$, where $e = 1$ is to be taken in the first case and $e = -1$ in the second. Then ${}_{v}\kappa_{n}$ is the magnitude of the normal component in (2.1).

Definition 1. The signed magnitude of the normal curvature vector of a vector field with respect to a curve at P is called *the normal curvature of the vector field with respect to the curve* at P. It will be denoted by $e({}_{v}\kappa_{n})$.

The normal curvature of the field v with respect to C is defined more generally by

$$e({}_{v}\kappa_{n}) = d_{\delta\gamma}p^{\delta}du^{\gamma}/(g_{\alpha\gamma}du^{\alpha}du^{\gamma}g_{\lambda\mu}p^{\lambda}p^{\mu})^{\frac{1}{2}} \tag{2.2}$$

when p^{δ} are not necessary the components of a unit vector.

The normal curvatures of v with respect to all curves through P on S are equal if and only if ${}_{v}\kappa_{n}$ is independent of du^{α}. Thus $\partial({}_{v}\kappa_{n})/\partial(du^{\alpha}) = 0$ for all du^{α} at P, which can be found to be equivalent to $gd = 0$ at P. Since $g > 0$, and (2.2) is symmetric in p^{α} and du^{α}, we conclude that the normal curvature of any vector field with respect to any curve in the surface is a constant at a point if and only if $d = 0$ at the point. This constant is equal to zero by Definition 3 which follows. Unless otherwise indicated, we shall always assume $d \neq 0$. It is easy to show that there exists a unique direction with respect to which the normal curvature of a vector field at P has a finite extreme value different from zero.

Definition 2. The unique direction, with respect to which the normal curvature of a vector field at P has its extreme value, is known as *the principal direction of the field* at P and the corresponding normal curvature is called *the principal curvature of the field* at P. A curve on S, whose direction at each and every point is a principal direction of a field, is called *a line of curvature of the field.*

The principal direction of the vector field v and the line of curvature of the vector field v at P are found from (2.2) by setting $\partial({}_{v}\kappa_{n})/\partial(du^{\alpha})$ equal to zero. The resulting equation after simplification is

$$\epsilon^{\alpha\beta}g_{\alpha\gamma}d_{\beta\delta}p^{\delta}du^{\gamma} = 0. \tag{2.3}$$

Definition 3. A direction, with respect to which the normal curvature of a vector field is zero, is called an *asymptotic direction of the vector field.* A curve on the surface, whose direction at each and every point is an asymptotic direction of a vector field, is called *an asymptotic line of the vector field.*

The asymptotic direction of the vector field v at P and the asymptotic line of the vector field v at P are defined by the equation

$$d_{\delta\gamma}p^{\delta}du^{\gamma} = 0. \tag{2.4}$$

Definition 4. A curve on the surface, along which the vectors of a vector field of S are tangent to the curve, is called *a curve of the vector field.*

The curve of the vector field v at P is defined by

$$\epsilon_{\alpha\beta}p^{\alpha}du^{\beta} = 0. \tag{2.5}$$

The normal curvature of a vector field with respect to the curve of the field at a point is the normal curvature of the curve at the point. In this respect, the normal curvature of a curve on the surface may be considered as a special case of the normal curvature of a vector field.

Definition 5. A curve on the surface, relative to which the vectors of a given field are parallel in the sense of Levi-Civita, is called *an indicatrix of the vector field.*

The indicatrix of the vector field v at P is defined by

$$p^{\alpha}_{,\gamma}du^{\gamma} = 0. \tag{2.6}$$

Definition 6. A field of vectors, whose directions at each and every point of S constitute a family of principal directions of S, is called *a principal vector field* or *a principal field of* S. The direction of the vector of a principal field and the line of curvature of S at P determined by the principal field are respectively called *the corresponding principal direction of* S and *the corresponding line of curvature of* S *at* P.

It is obvious that the curve of a principal field at P is the corresponding line of curvature of S at P. The vector field v is a principal field of S if and only if

$$\epsilon^{\alpha\beta}d_{\alpha\gamma}g_{\rho\beta}p^{\gamma}p^{\delta} = 0. \tag{2.7}$$

3. Properties. From Definitions 3 and 4, it is obvious that the asymptotic direction of a vector field is conjugate to the direction of the vector of

the field at P. The orthogonal trajectory of the asymptotic line of the vector field v at P is given by the equation

$$\epsilon_{\alpha\beta}g^{\alpha\gamma}d_{\delta\gamma}p^{\delta}du^{\beta}=0,$$

which is the same as (2.3) since $\epsilon_{\alpha\beta}g^{\alpha\gamma}=\epsilon^{\lambda\gamma}g_{\lambda\beta}$. Hence we have

THEOREM 3.1. *The asymptotic lines of a vector field form a conjugate net with the curves of the field on the surface and form an orthogonal net with the lines of curvature of the field on the surface.*

Let v be a principal vector field. Then on substituting for p^{α} in (2.7) their expressions from equation (2.3), we obtain

$$\epsilon^{\beta\gamma}g_{\delta\gamma}d_{\alpha\beta}e^{\alpha\mu}\epsilon^{\lambda\nu}g_{\lambda\sigma}d_{\nu\mu}du^{\sigma}e^{\delta\rho}\epsilon^{\psi\omega}g_{\psi\tau}d_{\omega\rho}du^{\tau}=0. \tag{3.1}$$

By the application of the process of contraction and the formulae

$$\epsilon^{\alpha\beta}=e^{\alpha\beta}/g^{\frac{1}{2}},\qquad \epsilon_{\delta\alpha}d^{\delta\beta}=\epsilon^{\sigma\beta}d_{\sigma\alpha},\qquad d^{\alpha\beta}d_{\beta\gamma}=\delta^{\alpha}{}_{\gamma},$$

$$\epsilon^{\alpha\beta}g_{\alpha\gamma}g_{\beta\delta}=\epsilon_{\gamma\delta},\qquad \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}d_{\alpha\beta}=d^{\gamma\delta},$$

equation (3.1) can be reduced to the form $\epsilon^{\alpha\psi}d_{\alpha\sigma}g_{\tau\psi}du^{\sigma}du^{\tau}=0$, which is the equation of the lines of curvature of S. Since p^{α} can not be orthogonal to du^{α} by Theorem 3.1 and by the assumption $d\neq 0$, they must be coincident.

THEOREM 3.2. *The principal direction of a principal vector field of S at P coincides with the corresponding principal direction of S at P. The line of curvature of a principal vector field of S at P is the corresponding line of curvature of S at P. The principal curvature of a principal vector field at P is the corresponding principal curvature of S at P.*

The principal direction of the vector field v at P is found from (2.3) to be

$$du^{\gamma}=g^{\frac{1}{2}}g^{\gamma\beta}d_{\delta\beta}p^{\delta}. \tag{3.2}$$

The square of the length of this vector is equal to

$$g_{\alpha\beta}du^{\alpha}du^{\beta}=gg^{\beta\lambda}d_{\delta\beta}d_{\mu\lambda}p^{\delta}p^{\mu}. \tag{3.3}$$

Substitution of (3.2), (3.3) into (2.2) gives

$$e(_{v}\bar{\kappa}_{n})=(h_{\lambda\delta}p^{\lambda}p^{\delta}/g_{\lambda\delta}p^{\lambda}p^{\delta})^{\frac{1}{2}} \tag{3.4}$$

where $h_{\alpha\beta}=\sum_{i}(\partial X^{i}/\partial u^{\alpha}\cdot\partial X^{i}/\partial u^{\beta})$.

The extreme values of the extreme values (3.4) at P as the vector field v varies are given by those vector fields v for which $\partial(_{v}\bar{\kappa}_{n})/\partial p^{\delta}=0$, that is,

$g_{\alpha\beta}p^\alpha p^\beta h_{\lambda\delta}p^\lambda - h_{\alpha\beta}p^\alpha p^\beta g_{\delta\lambda}p^\lambda = 0$. By means of (3.4) this equation is expressible in the form

$$(h_{\delta\lambda} - {}_v\bar{\kappa}_n{}^2 g_{\delta\lambda})\,p^\lambda = 0. \tag{3.5}$$

Eliminating ${}_v\bar{\kappa}_n{}^2$ from (3.5) we obtain $\begin{vmatrix} h_{1\lambda}p^\lambda & h_{2\lambda}p^\lambda \\ g_{1\lambda}p^\lambda & g_{2\lambda}p^\lambda \end{vmatrix} = 0$, which, in consequence of [3, p. 253], implies that

$$h_{\alpha\beta} = d_{\alpha\beta}M - g_{\alpha\beta}K, \text{ i. e., } \begin{vmatrix} d_{1\lambda}p^\lambda & d_{2\lambda}p^\lambda \\ g_{1\lambda}p^\lambda & g_{2\lambda}p^\lambda \end{vmatrix} = 0, \tag{3.6}$$

where M and K are respectively the mean curvature and the Gaussian curvature of S. Hence p^α are the components of the principal directions of S.

THEOREM 3.3. *The two principal vector fields of S are the vector fields in S which have the extremal principal curvatures of all vector fields in S at P.*

This theorem characterizes a line of curvature of S as a curve whose unit tangent vectors at each and every point form a vector field whose principal curvature assumes an extreme value of the normal curvatures of all vector fields in S.

From Theorems 3.2 and 3.3 it is obvious that the extreme values of the principal curvatures of all vector fields in S are equal respectively to the principal curvatures of the surface. Hence

THEOREM 3.4. *The principal curvatures of the surface at a point are the extremal principal curvatures of all vector fields in the surface at the point.*

Since $h_{\alpha\beta}$ is the fundamental tensor of the Gaussian representation G of the surface S, $(h_{\alpha\beta}p^\alpha p^\beta)^{\frac{1}{2}}$ is the magnitude of the vector p^α in G. If we assume v to be a field of unit vectors in S, then (3.4) reduces to

$$e({}_v\bar{\kappa}_n) = (h_{\alpha\beta}p^\alpha p^\beta)^{\frac{1}{2}}. \tag{3.7}$$

Hence we have

THEOREM 3.5. *The principal curvature of a unit vector field in S at P is numerically equal to the magnitude of the same vector in the Gaussian representation of S at the corresponding point.*

If the vector field does not necessary consist of unit vectors, the content of the above theorem can be stated as follows:

The principal curvature of a vector field in S at P is numerically equal to the ratio of the two magnitudes of the same vector in the Gaussian representation of S and in S at the corresponding points.

If equations (3.6) be multiplied by $p^\alpha p^\beta$ and summed up with respect to α and β, we have

$$(3.8) \qquad ({}_v\bar{\kappa}_n)^2 = M\kappa_n - K,$$

where κ_n is the normal curvature of the curve of the field v on S. Hence

THEOREM 3.6. *The square of the principal curvature of a vector field in S at P is a number diminished by the Gaussian curvature of S at P, the number being the product of the mean curvature of S and the normal curvature of the curve of the field at P.*

Let p^α and q^α be two orthogonal vector fields. Let $e({}_p\bar{\kappa}_n)$ and $e({}_q\bar{\kappa}_n)$ denote respectively the principal curvatures of the two fields. Let κ_1 and κ_2 be the principal curvatures of the surface S. Then by (3.8) and by Euler's theorem we have

$$(3.9) \qquad ({}_p\bar{\kappa}_n)^2 + ({}_q\bar{\kappa}_n)^2 = M^2 - 2K = \kappa_1^2 + \kappa_2^2.$$

Hence

THEOREM 3.7. *The sum of the squares of the principal curvatures of two orthogonal vector fields at P is constant and is independent of the choice of the two orthogonal vector fields. The constant is equal to the square of the mean curvature of S diminished by twice of the Gaussian curvature of S at P or equal to the sum of the squares of the principal curvatures of S at P.*

Similarly, from (3.8) we find

$$(3.10) \qquad ({}_p\bar{\kappa}_n \cdot {}_q\bar{\kappa}_n)^2 = K^2 + \tfrac{1}{4}M^2(\kappa_1 - \kappa_2)^2 \sin^2 2\theta,$$

where θ is the angle between a line of curvature of S at P and the curve of one of the two fields at P.

By Theorem 3.4 it is evident that if P is an elliptic point, the principal curvatures of all vector fields at P are of the same sign. If P is a hyperbolic point, the principal curvatures of certain vector fields are positive and the principal curvatures of other vector fields are negative.

At a hyperbolic point the normal curvatures of a vector field with respect to different curves are either all positive or all negative, because the normal curvature of a vector field is zero with respect to the asymptotic line of the field, which is orthogonal to the principal direction of the field, and because ${}_v\kappa_n$ is continuous at every point of S. Hence the principal curvature of a vector field at either an elliptic point or a hyperbolic point is sufficient for the determination of the sign of the normal curvatures of the vector field with respect to different curves. As a vector field at the hyperbolic point P

varies from one principal direction of S to another principal direction of S through an asymptotic direction of S, the principal curvature of the field changes from $+$ to $-$ or vice versa. Hence the principal curvature of a vector field has the sign of that principal curvature of S which is situated together with the vector of the field at P in the same section separated by the asymptotic lines of S at P. Consequently, at a hyperbolic point of S the principal curvatures of the two orthogonal vector fields may or may not differ in sign. From (3.10) we have

THEOREM 3.8. *At an elliptic or hyperbolic point P of S, the product of the principal curvatures of two orthogonal vector fields is respectively $\geqq$ or numerically $\geqq$ the Gaussian curvature of S at P. The equality sign holds if and only if P is an umbilic point of S, or the two vector fields are the principal fields of S, or S is a sphere or a minimal surface.*

Let $\lambda_1|^\alpha$ be a unit vector in the asymptotic direction of the vector field v at P and $\lambda_2|^\alpha$ a unit vector in the principal direction of the vector field v at P on S. Then we have $g_{\alpha\beta}\lambda_1|^\alpha\lambda_2|^\beta = 0$, $g_{\alpha\beta}\lambda_\gamma|^\alpha\lambda_\gamma|^\beta = 1$; $\gamma = 1, 2$. Any unit vector tangential to S at P, say du^α, can be expressed as a linear combination of $\lambda_1|^\alpha$ and $\lambda_2|^\alpha$, such as

$$du^\alpha = \lambda_1|^\alpha \cos\theta_1 + \lambda_2|^\alpha \cos\theta_2, \tag{3.11}$$

where $\cos\theta_\gamma = g_{\alpha\beta}\lambda_\gamma|^\alpha du^\beta$; $\gamma = 1, 2$. Assuming $g_{\alpha\beta}p^\alpha p^\beta = 1$ and making use of (2.2) and Definitions 2 and 3, we find the normal curvature of the vector field v with respect to a curve with (3.11) as its tangent vector at P is equal to $e({}_v\kappa_n) = e({}_v\bar{\kappa}_n)\cos\theta_2$. Thus the normal curvature of a vector field with respect to an arbitrary curve can be expressed in terms of the principal curvature of the field. Hence we have the following property similar to Euler's and Meusnier's theorems.

THEOREM 3.9. *The normal curvature of a vector field with respect to an arbitrary curve C always satisfies the following first relation with the principal curvature of the field and the second relation with the absolute curvature of the field with respect to C:*

$${}_v\kappa_n = {}_v\bar{\kappa}\cos\phi, \qquad {}_v\kappa_n = {}_v\kappa\cos\omega,$$

where ϕ is the angle between the direction of the curve C and the principal direction of the field and ω denotes the angle between w^i and X^i at the point considered.

Let a vector field in S be given. Through a point P on S, there pass the following two curves: the curve of the field and the asymptotic line of the field. The straight lines along these two curves in the direction of the vectors of the field associated to the curve form generators of two developable surfaces. The first one is the tangential surface of the curve of the field and the second one is developable because of the conjugacy between the asymptotic direction of the field and the vector of the field. We want to show that these two curves are the only curves possessing this property on S.

Let C: $u^\alpha = u^\alpha(s)$ be a curve on S such that the straight lines on v associated with C are generators of a developable surface. Then the developable surface will be defined by

$$y^i = x^i(s) + \lambda v^i, \tag{3.12}$$

where s and λ are parameters. $\lambda = a$ constant $\neq 0$ defines a curve on the developable surface, the tangent of which has the direction defined by $\partial y^i/\partial s = t^i + \lambda \cdot {}_v k$, where ${}_v k$ has the same meaning as in (2.1).

Since $(t^i, v^i, t + \lambda \cdot {}_v k) = 0$ implies $(t^i, v^i, {}_v k) = 0$, we find that a necessary and sufficient condition for the surface (3.12) to be a developable is ${}_v k = 0$ or $t^i = \pm v^i$.

Case 1. ${}_v k = 0$. Then $v^i = \text{const.}$ The vectors of the field are absolutely parallel. The lines on them form generators of a developable surface along any curve on the surface. The surface is consequently isometric with the plane. If $v^i \neq \text{const.}$, then ${}_v \kappa = 0$ and from (2.1) we have $p^\alpha_{;\gamma} du^\gamma = 0$, $p^\delta d_{\delta\gamma} du^\gamma = 0$, which mean geometrically that the lines on the vectors of the field will form generators of a developable surface along the curve C which is both an indicatrix of the field and an asymptotic line of the field.

Case 2. $t^i = \pm v^i$. The curve C is then a curve of the vector field. Hence

THEOREM 3.10. *At a point on a non-developable surface there are two and only two curves, along each of which the straight lines on the vectors of a field form generators of a developable surface. These curves are the asymptotic line of the field and the curve of the field.*

The following theorem is an immediate consequence of Definitions 1 and 5:

THEOREM 3.11. *The normal curvature of a vector field at P is the same with respect to all curves tangent at P to the same direction. The normal curvature of either one of two vector fields with respect to the curve of the other field is the same. The absolute curvature vector of a vector field with respect to the indicatrix of the field has at each point the direction of the surface normal. The absolute curvature and the normal curvature of a vector field with respect to a curve C are numerically equal if and only if the curve C is the indicatrix of the field.*

4. Generalization. Let $V_n: y^\alpha = y^\alpha(x^1, \cdots, x^n)$, $\alpha = 1, \cdots, n+1$ be a hypersurface in V_{n+1}. Let $C: x^i = x^i(s)$, $i = 1, \cdots, n$ be a curve on V_n. Let v be a field of unit vectors in V_n such that $v^\alpha = p^i y^\alpha_{,i}$ and v^α are real analytic functions of x^i. Then *the normal curvature of v with respect to C at a point P of V_n* is defined by

$$e({}_v\kappa_\xi) = \Omega_{ij}p^i dx^j / (g_{ij}dx^i dx^j g_{hl}p^h p^l)^{\frac{1}{2}}. \tag{4.1}$$

We assume the fundamental forms of all spaces discussed in this section are positive definite.

When, in particular, the vector v is tangent to C at P,

$$e({}_v\kappa_\xi) = \Omega_{ij}dx^i dx^j / g_{ij}dx^i dx^j$$

and, consequently, the normal curvature of v with respect to C becomes the normal curvature of the curve C. It is evident ${}_v\kappa_\xi = 0$ at P along the direction for which

$$\Omega_{ij}p^i dx^j = 0 \tag{4.2}$$

This direction is called *an asymptotic direction of the vector field v at P of V_n.* A curve, whose direction at each and every point is an asymptotic direction of a vector field is *an asymptotic line of the field.* Such a direction and such a line satisfy the differential equation (4.2).

The extreme values of $e({}_v\kappa_\xi)$ at P are attained with respect to those directions for which $\partial({}_v\kappa_\xi)/\partial(dx^i) = 0$ for $i = 1, \cdots, n$, that is

$$(\Psi_{ij} - {}_v\kappa_\xi^2 g_{ij})dx^i = 0, \tag{4.3}$$

where

$$\Psi_{ij} = \Omega_{ij}\Omega_{hl}p^h p^l / g_{ij}p^i p^j \tag{4.4}$$

and where ${}_v\kappa_\xi^2$ is a root of the equation

$$|\Psi_{ij} - {}_v\kappa_\xi^2 g_{ij}| = 0. \tag{4.5}$$

Since the metric of V_n is assumed to be positive definite, the roots of (4. 5) are all real; and since $\| \Psi_{ij} \|$ is of rank 1, $n-1$ roots are equal to zero which correspond to asymptotic directions of the vector field v at P. The square of the non-zero extreme value corresponds by (4. 3) to a principal direction determined by the tensor Ψ_{ij}. The extreme value of $e({}_v\kappa\xi)$ is called *the principal normal curvature of the vector field v at P in* V_n and the corresponding direction is *the principal direction of the field at P in* V_n.

From (4. 1) we know that *the normal curvature of a vector field in* V_n *has equal value with respect to all curves of* V_n *through P with the same direction. Its value with respect to an indicatrix of the field is the absolute curvature of the field in* V_{n+1} *with respect to the indicatrix of the field of* V_n, since *an indicatrix of the field v* satisfies $p^i_{,j}dx^j = 0$, and ${}_v\kappa$, the *absolute curvature of the field v with respect to C*, is defined by

$$ {}_vk = {}_v\kappa w^\alpha = dv^\alpha/ds = p^i_{,j}y^\alpha_{,i}dx^j/ds + e({}_v\kappa\xi)\xi^\alpha, $$

where w^α is a unit vector in the same sense as ${}_vk$. Similarly, *when an asymptotic line* (4. 2) *of a vector field v is an indicatrix of the field in* V_n, *the latter is also the indicatrix of the field in the enveloping space* V_{n+1} *and conversely.*

The principal direction of the vector field v in V_n *is indeterminate if and only if*

$$ \Psi_{ij} = {}_v\kappa\xi^2 g_{ij}. \tag{4. 6} $$

Such space is then homogeneous with respect to the tensor Ψ_{ij}. *An asymptotic line of a vector field is conjugate to a curve of the field at P in* V_n.

If the vector field v is a principal vector field in V_n, then

$$ (\Omega_{ij} - \kappa\xi g_{ij})p^i = 0, \tag{4. 7} $$

where $\kappa\xi$ is the corresponding principal curvature of V_n at P. In virtue of (4. 1), (4. 3), (4. 4) and 4. 7) we obtain $e({}_v\kappa\xi) = \kappa\xi$ at P. It follows that:

The principal direction of a principal vector field in V_n *at P and the principal curvature of the same field in* V_n *at P are respectively identical with the corresponding principal direction of* V_n *and the corresponding principal curvature of* V_n *at P.* Consequently, *a line of curvature of a principal vector field in* V_n *is the corresponding line of curvature of* V_n.

Let the unit vectors of an orthogonal ennuple at P formed by $\lambda_h|^i$ be taken in the principal direction of the vector field v at P in V_n and

asymptotic directions of the vector field v at P in V_n. In particular, let $\lambda_1|^i$ be in the principal direction. Then we have

$$
\begin{aligned}
&g_{ij}\lambda_h|^i\lambda_h|^j = 1, \qquad g_{ij}\lambda_h|^i\lambda_k|^j = 0, \qquad h \neq k \\
&\Psi_{ij}\lambda_h|^i\lambda_h|^j = {}_v\kappa\xi^2, \qquad \Psi_{ij}\lambda_h|^i\lambda_k|^j = 0, \qquad h \neq k.
\end{aligned} \tag{4.8}
$$

Any field of unit vectors in V_n, say dx^i, is defined by

$$
dx^i = \lambda_1|^i \cos\theta_1 + \cdots + \lambda_n|^i \cos\theta_n, \tag{4.9}
$$

where $\cos\theta_r = g_{ij}\lambda_r|^i dx^j$. Thinking that p^i and dx^i are unit vectors, we can write (4.1) in the form $e({}_v\kappa\xi) = \Omega_{ij}p^i dx^j$, and by (4.4) we have

$$
({}_v\kappa\xi)^2 = \Psi_{ij}dx^i dx^j. \tag{4.10}
$$

Substituting in (4.10) from (4.9) and making use of (4.3) and (4.8), *we obtain*

$$
{}_v\kappa\xi^2 = {}_v\bar{\kappa}\xi^2 \cos^2\theta_1 \tag{4.11}
$$

which is analogous to Euler's formula, $e({}_v\bar{\kappa}\xi)$ being the principal curvature of the vector field.

Consider a Riemannian space V_n of coordinates x^i immersed in a Riemannian space V_m of coordinates y^α. Then *the normal curvature of the vector field v at P in V_n in V_m* is denoted by $e({}_v\kappa\xi)$ such that

$$
({}_v\kappa\xi)^2 = \Big(\sum_\nu \Omega_\nu|_{ij}\Omega_\nu|_{kl}p^i p^k dx^j dx^l\Big)/(g_{ij}g_{kl}p^i p^j dx^k dx^l), \tag{4.12}
$$

where $i, j, k, l = 1, \cdots, n$, $\nu = n+1, \cdots, m$. Proceeding with this definition and others in a manner similar to the preceding discussion for V_n in V_{n+1}, we may obtain for V_n in V_m the following properties:

The normal curvature at a point P of a vector field with respect to a curve C in V_n in V_m depends only on the direction dx^i of C at P, and is the same with respect to all curves tangent to C at P in V_n. It is zero in an asymptotic direction of the field, which is conjugate to the vector of the field, and its finite extreme value for the normal $\xi_\nu|^\alpha$ is attained with respect to the principal direction of the field for the normal $\xi_\nu|^\alpha$. The principal direction at P of a principal vector field in V_n and the principal normal curvature at P of the same field for the normal $\xi_\nu|^\alpha$ are respectively identical with the corresponding principal direction of V_n at P and the corresponding curvature of V_n at P for the normal $\xi_\nu|^\alpha$. A line of curvature of a principal vector field in V_n for the normal $\xi_\nu|^\alpha$ is the line of curvature determined by this

corresponding family of principal directions of V_n for the normal $\xi_\nu|^\alpha$. The normal curvature of a vector field with respect to an indicatrix of the field in V_n. An asymptotic line of a vector field in V_n and an indicatrix of the field in V_n are coincident if and only if the latter is also an indicatrix of the field in V_m.

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A SYNTHESIS OF THE CLIFFORD MATRICES AND ITS GENERALIZATION.* [1]

By T. G. Room.

Clifford matrices of 2^r row and columns are a set of $2r+1$ matrices γ_i, with the properties

$$\gamma_i^2 = I, \qquad \gamma_i\gamma_j + \gamma_j\gamma_i = 0.$$

In Part I these matrices are synthesized from two elementary sets of r matrices, U_i which are diagonal matrices, and V_i which are substitution matrices. The key to the development is the numbering of the rows and columns of the matrix in the reversed binary scale, namely as $\alpha_1 + 2\alpha_2 + \cdots + 2^{r-1}\alpha_r$.

In Part II, geometrical properties of the collineations determined by these matrices are worked out, and in particular the existence of a sequence of systems of spaces exhibiting a "Mutuality" exactly analogous to Study's "Triality" is established.

Part III suggests generalizations of Part I using scales other than the binary to obtain sets of matrices in which most pairs commute, but for some pairs, in one case $UV = kVU$, and in another $U^2V = VU$.

Part I.

1. The basic family of matrices. Number the rows and columns of a $2^r \times 2^r$ matrix in the reversed binary scale, beginning with 0, so that the row or column in position number $\alpha_1 + 2\alpha_2 + 4\alpha_3 + \cdots + 2^{r-1}\alpha_r$ from the top or the left carries the indices $\alpha_1\alpha_2\alpha_3\cdots\alpha_r$. Throughout Parts I and II of this paper I use the conventions

(i) Greek indices take the values 0 and 1 only,

(ii) All arithmetical functions of Greek indices are reduced modulo 2,

(iii) For any index α_i, α'_i is defined by $\alpha_i + \alpha'_i = 1$.

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[1] My interest in the Clifford matrices was aroused by Professor O. Veblen, and Drs. W. J. Givens and N. H. Kuiper. A recently published paper by Dr. Kuiper (1949) gives much of the geometrical background, and many references to the history of the problem may be found in it. Part II of this paper was presented to the International Congress of Mathematicians.

Denote the elements of a matrix by $a^{\alpha_1\alpha_2\ldots\alpha_r}_{\beta_1\beta_2\ldots\beta_r}$ or $a^{(\alpha_i)}_{(\beta_j)}$ (with superscript as row number) and define matrices U_i, V_i thus:

U_i is the diagonal matrix with elements

$$a^{\alpha_1\alpha_2\ldots\alpha_r}_{\alpha_1\alpha_2\ldots\alpha_r} = (-1)^{\alpha_i}$$

V_i is the substitution matrix with only one non-zero element in each row and column, given by

$$a^{\alpha_1\alpha_2\ldots\alpha_{i-1}\alpha_i\alpha_{i+1}\ldots\alpha_r}_{\alpha_1\alpha_2\ldots\alpha_{i-1}\alpha'_i\alpha_{i+1}\ldots\alpha_r} = 1,$$

where each superscript except the i-th is equal to the corresponding subscript, and the i-th superscript and subscript are different.

For $r = 2$ the matrices are

$$U_1 = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}, \qquad U_2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \qquad V_2 = \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}.$$

These matrices have the properties:

THEOREM 1.1. (i) $U_i^2 = V_i^2 = I$,

(ii) *All pairs commute, except the r pairs* U_i, V_i,

(iii) $U_iV_i = -V_iU_i$.

THEOREM 1.2. $W^{(\alpha_i)}_{(\beta_j)}$, $= \Pi U_i^{\alpha_i} \Pi V_j^{\beta_j}$, *is the matrix with one non-zero element in each row and column, given by*

$$a^{(\lambda_i)}_{(\lambda_i+\beta_j)} = (-1)^{\Sigma\alpha_i\lambda_i}.$$

E. g., when $r = 2$,

$$W^{11}_{01} = U_1U_2V_2 = \begin{bmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{bmatrix}.$$

THEOREM 1.3. $$W^{(\alpha_i)}_{(\beta_j)} W^{(\gamma_i)}_{(\delta_j)} = (-1)^{\Sigma\beta_j\gamma_j} W^{(\alpha_i+\gamma_i)}_{(\beta_j+\delta_j)}.$$

THEOREM 1.4. $[\boldsymbol{W}^{(\alpha_i)}_{(\beta_i)}]^2 = (-1)^{\Sigma\alpha_i\beta_i}\boldsymbol{I}$.

If we take $\boldsymbol{W}^{(\alpha_j)}_{(\beta_k)*} = i^{\Sigma\alpha_j\beta_j}\boldsymbol{W}^{(\alpha_j)}_{(\beta_k)}$, $\Sigma\alpha_j\beta_j$ being reduced modulo 2, then

$$[\boldsymbol{W}^{(\alpha_j)}_{(\beta_k)*}]^2 = I.$$

THEOREM 1.5. *Two matrices* $\boldsymbol{W}^{(\alpha_i)}_{(\beta_j)}$, $\boldsymbol{W}^{(\gamma_i)}_{(\delta_j)}$ *commute or anti-commute according as* $\Sigma\alpha_i\delta_i + \Sigma\beta_j\gamma_j = 0$, *or* 1.

THEOREM 1.6. $\boldsymbol{W}^{(\alpha_i)}_{(\beta_i)}$ *is symmetric or skew-symmetric according as* $\Sigma\alpha_i\beta_i = 0$, *or* 1.

Thus, the non-zero elements of the matrix are

$$a^{(\lambda_i)}_{(\lambda_j+\beta_j)} = (-1)^{\Sigma\alpha_i\lambda_j}.$$

Since the element $a^{(\lambda_i)}_{(\lambda_i+\beta_i)} = a^{(\lambda_i+\beta_i)}_{(\lambda_i+\beta_i+\beta_i)}$, it is also a non-zero element of the matrix, and

$$a^{(\lambda_i+\beta_i)}_{(\lambda_i)} = (-1)^{\Sigma\alpha_i(\lambda_i+\beta_i)} = (-1)^{\Sigma\alpha_i\beta_i}a^{(\lambda_i)}_{(\lambda_i+\beta_i)}.$$

THEOREM 1.7. *Any given* $2^r \times 2^r$ *matrix,* $\boldsymbol{H} = (h^{(\alpha_i)}_{(\beta_j)})$, *may be expressed in a single way as* $\Sigma k^{(\alpha_i)}_{(\beta_j)}\boldsymbol{W}^{(\alpha_i)}_{(\beta_j)}$, *there being* 2^{2r} *terms in the summation.*

Thus, if we rearrange the terms in the rows of the matrix $\boldsymbol{H}$, so that they form the matrix $\tilde{\boldsymbol{H}} = (h^{(\alpha_i)}_{(\alpha_j+\beta_j)})$, then $\tilde{\boldsymbol{H}} = \boldsymbol{\Omega K}$ where $\boldsymbol{K} = (k^{(\alpha_i)}_{(\beta_i)})$ and $\boldsymbol{\Omega} = \Sigma\boldsymbol{W}^{\alpha_1\alpha_2\ldots\alpha_r}_{\alpha'_1\alpha'_2\ldots\alpha'_r}$, that is, $\boldsymbol{\Omega}$ is the matrix each of whose elements is $+1$ or -1, the sign being given by $a^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_r} = (-1)^{\Sigma\alpha_i\beta_i}$.

For $r = 2$, we have

$$\begin{bmatrix} h^{00}_{00} & h^{00}_{10} & h^{00}_{01} & h^{00}_{11} \\ h^{10}_{10} & h^{10}_{00} & h^{10}_{11} & h^{10}_{01} \\ h^{01}_{01} & h^{01}_{11} & h^{01}_{00} & h^{01}_{10} \\ h^{11}_{11} & h^{11}_{01} & h^{11}_{10} & h^{11}_{00} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} k^{00}_{00} & k^{00}_{10} & k^{00}_{01} & k^{00}_{11} \\ k^{10}_{00} & k^{10}_{10} & k^{10}_{01} & k^{10}_{11} \\ k^{01}_{00} & k^{01}_{10} & k^{01}_{01} & k^{01}_{11} \\ k^{11}_{00} & k^{11}_{10} & k^{11}_{01} & k^{11}_{11} \end{bmatrix}$$

$$\boldsymbol{\Omega} = \boldsymbol{U}_1\boldsymbol{U}_2 + \boldsymbol{V}_1\boldsymbol{U}_2 + \boldsymbol{U}_1\boldsymbol{V}_2 + \boldsymbol{V}_1\boldsymbol{V}_2.$$

2. The Clifford matrices. The Clifford matrices $\boldsymbol{\gamma}_i^{(r)}$, of 2^r rows and columns, have the properties:

$$\boldsymbol{\gamma}_i^2 = \boldsymbol{I}, \qquad \boldsymbol{\gamma}_i\boldsymbol{\gamma}_j + \boldsymbol{\gamma}_j\boldsymbol{\gamma}_i = \boldsymbol{O}.$$

They may be defined inductively from the matrix $\gamma_0^{(0)}$ of one element thus:

$$\gamma_0^{(0)} = (1); \quad \gamma_j^{(r)} = \begin{pmatrix} \gamma_j^{(r-1)} & \\ & -\gamma_j^{(r-1)} \end{pmatrix},$$

$$\gamma_{2r-1}^{(r)} = i \begin{pmatrix} & I \\ -I & \end{pmatrix}, \quad \gamma_{2r}^{(r)} = \begin{pmatrix} & I \\ I & \end{pmatrix}.$$

In the notation used in this paper, this set is

$$\gamma_0^{(r)} = W_{00\ldots0}^{11\ldots1}, \qquad \gamma_1^{(r)} = iW_{100\ldots0}^{11\ldots1}, \qquad \gamma_2^{(r)} = W_{100\ldots0}^{011\ldots1},$$

$$\gamma_3^{(r)} = iW_{010\ldots0}^{011\ldots1}, \qquad \gamma_4^{(r)} = W_{0100\ldots0}^{0011\ldots1}, \cdots,$$

$$\gamma_{2r-1}^{(r)} = iW_{000\ldots01}^{000\ldots01}, \qquad \gamma_{2r}^{(r)} = W_{00\ldots01}^{00\ldots00};$$

that is: $\gamma_{2k}^{(r)} = W_{(\beta_j)}^{(\alpha_i)}$, where $\alpha_1, \cdots, \alpha_k = 0$, $\alpha_{k+1}, \cdots, a_r = 1$, and all $\beta_j = 0$, except $\beta_k = 1$, and $\gamma_{2k+1}^{(r)} = iW_{(\beta_j)}^{(\alpha_i)}$ where $\alpha_1, \cdots, \alpha_k = 0$, $\alpha_{k+1}, \cdots, \alpha_r = 1$, and all $\beta_j = 0$, except $\beta_{k+1} = 1$.

For our imemdiate purpose the factor i is unimportant, and we define as the *primary Clifford set*, the set of $2r+1$ matrices W which correspond to the matrices $\gamma^{(r)}$. A *Clifford set* of matrices W is one with the same properties as the $\gamma^{(r)}$, namely $W^2 = \pm I$, $W_jW_k = -W_kW_j$, that is, a set of matrices W such that no pair commutes.

One of the problems (the solution of which I do not attempt here) is the determination of all Clifford sets. It is clear that from any given set we may obtain other sets by any of the operations

(i) changing each U into a V and each V into a U, i. e., by replacing each $W_{(\beta_j)}^{(\alpha_i)}$ by $W_{(\alpha_i)}^{(\beta_j)}$. E. g., the second set of those listed below is obtained in this way from the first,

(ii) effecting on each symbol of the given set the same permutation of both superscripts and subscripts. E. g., the third set below is obtained from the first by interchanging the first and second indices,

(iii) adjoining I to the set, multiplying by any member of the set, and removing I from the resulting set. E. g., the second set below is obtained from the first by multiplying by W_{01}^{01}.

From Theorem 1.5 it follows that the problem is equivalent to the following combinatorial problem:

Let $(\alpha_i^{(s)}; \beta_j^{(s)})$, $s = 1, \cdots, m$, be m pairs of permutations of r terms

each 0 or 1. It is required to determine all sets of m such pairs of permutations, which are such that for every two pairs in a set

$$\sum_i \alpha_i^{(s)}\beta_i^{(t)} + \sum_i \alpha_i^{(t)}\beta_i^{(s)} \text{ is odd.}$$

In the solutions above $m = 2r + 1$, and this is probably the largest value of m.

In the case $r = 2$, there are six Clifford sets, namely

$$\begin{array}{lllll}
W_{00}^{11} & W_{10}^{11} & W_{10}^{01} & W_{01}^{01} & W_{01}^{00} \\
W_{11}^{00} & W_{11}^{10} & W_{01}^{10} & W_{01}^{01} & W_{00}^{01} \\
W_{00}^{11} & W_{01}^{11} & W_{01}^{10} & W_{10}^{10} & W_{10}^{00} \\
W_{11}^{00} & W_{11}^{01} & W_{10}^{01} & W_{10}^{10} & W_{00}^{10} \\
W_{00}^{10} & W_{10}^{00} & W_{11}^{11} & W_{10}^{11} & W_{11}^{10} \\
W_{00}^{01} & W_{01}^{00} & W_{11}^{11} & W_{01}^{11} & W_{11}^{01}\,.
\end{array}$$

If these matrices are regarded as collineation matrices in projective [3], then: the five points obtained from any general point under the transformations of one of the sets lie in a plane and on a conic with the point, and the six such planes are the planes through a point of a Kummer 16_6 configuration.

Part II.

Quadrics associated with Clifford Matrices.

3. The Clifford involutions. For convenience of exposition, in this Part of the work I take $r = 4$.

As coordinates in [15] take $(x) = x_{\alpha\beta\gamma\delta}$,[2] where $\alpha, \cdots, \delta = 0, 1$, and $\alpha\beta\gamma\delta$ are the numbers 0 to 15 expressed as $\alpha + 2\beta + 4\gamma + 8\delta$. The primary Clifford matrices are

$$W_{0000}^{1111};\ W_{1000}^{1111},\ W_{1000}^{0111};\ W_{0100}^{0111},\ W_{0100}^{0011};\ \cdots;\ W_{0001}^{0001},\ W_{0001}^{0000}\,.$$

The pairs with identical subscripts have the non-zero elements, ± 1, in the same places, with half the signs agreeing. It is simpler therefore to replace these matrices by the set of 9:

---

[2] The coordinates $x_{\alpha\beta\gamma\delta}$ are related to Cartan's (Cartan, 1938, p. 4) coordinates $\xi_{ijk\ldots}$ thus: $i, j, k, \cdots$ are the position numbers in any order of the non-zero subscripts $\alpha, \beta, \gamma, \delta$. E. g., x_{1011} is the same as ξ_{134}.

$$Y = W^{1111}_{0000},\ Y^1_0 = \tfrac{1}{2}(W^{1111}_{1000} + W^{0111}_{1000}),$$
$$Y^1_1 = \tfrac{1}{2}(-W^{1111}_{1000} + W^{0111}_{1000}) = (Y^1_0)^T,$$
$$Y^2_0 = \tfrac{1}{2}(W^{0111}_{0100} + W^{0011}_{0100}), \cdots, Y^4_1 = \tfrac{1}{2}(-W^{0001}_{0001} + W^{0000}_{0001}).$$

Each of these has only 8 non-zero elements, which for Y_0 are above the leading diagonal and for Y_1 are below it. The non-zero elements of typical matrices are:

$$Y^1_0\colon\ a^{0\beta\gamma\delta}_{1\beta\gamma\delta} = (-1)^{\beta+\gamma+\delta}$$
$$Y^1_1\colon\ a^{1\beta\gamma\delta}_{0\beta\gamma\delta} = (-1)^{\beta+\gamma+\delta}$$
$$Y^2_0\colon\ a^{\alpha 0\gamma\delta}_{\alpha 1\gamma\delta} = (-1)^{\gamma+\delta}$$
$$Y^3_0\colon\ a^{\alpha\beta 0\delta}_{\alpha\beta 1\delta} = (-1)^{\delta}$$
$$Y^4_0\colon\ a^{\alpha\beta\gamma 0}_{\alpha\beta\gamma 1} = 1.$$

The nine matrices determine a linear family of matrices Γ of freedom 8 defined by

$$\Gamma = cY + \Sigma c^i_0 Y^i_0 + \Sigma c^i_1 Y^i_1.$$

From this it is easy to prove that

THEOREM 3.1. $(\Gamma)^2 = \mathscr{E} I$, *where* $\mathscr{E} = (c)^2 + \Sigma c^i_0 c^i_1$.

THEOREM 3.2. *If* $\mathscr{E} = 0$, Γ *is of rank* 8.

THEOREM 3.3. $\Gamma^T = cY + \Sigma c^i_1 Y^i_0 + \Sigma c^i_0 Y^i_1$.

The matrix Δ, $= W^{1010}_{1111}$,[3] which is the product of the four skew-symmetric matrices of the original set of 9 Clifford matrices, has the following relations to each Γ:

THEOREM 3.4. $\Delta\Gamma\Delta = \Gamma^T$.

THEOREM 3.5. $\Gamma^T\Delta\Gamma = \mathscr{E}\Delta$.

The first of these comes directly from relations such as

$$\Delta W^{1111}_{1000}\Delta = -W^{1111}_{1000},$$

[3] W^{1010}_{1111} is the reversed diagonal matrix with elements $a^{\alpha\beta\gamma\delta}_{\alpha'\beta'\gamma'\delta'} = (-1)^{\alpha+\gamma}$. The corresponding matrix associated with the set of $2r+1$ Clifford matrices of 2^r rows and columns is $W^{101010\ldots}_{111111\ldots}$, and is symmetrical or skew-symmetrical according as the number of non-zero superscripts is even or odd. I. e., the corresponding correlation is a polarity with regard to a quadric if $r = 4s-1,\ 4s$, and with regard to a null-system if $r = 4s+1,\ 4s+2$.

using the form above, and for the second we have

$$\mathbf{\Gamma}^T\mathbf{\Delta}\mathbf{\Gamma} = \mathbf{\Gamma}^T(\mathbf{\Gamma}^T\mathbf{\Delta}) = (\mathbf{\Gamma}^T)^2\mathbf{\Delta} = \mathscr{E}\mathbf{\Delta}.$$

If $\mathbf{\Gamma}$ is taken to be the matrix of a collineation, it follows from Theorem 3.1, that (i) the collineation is involutory, and (ii) if $\mathscr{E} = 0$, then the collineation is singular, and the invariant space is a [7], γ, say. From Theorem 3.5 it follows that the conjugate, $\mathbf{\Gamma}\boldsymbol{x}$, of a point $\boldsymbol{x}$, is such that $(\boldsymbol{x}^T\mathbf{\Gamma}^T)\mathbf{\Delta}\mathbf{\Gamma}\boldsymbol{x} = \mathscr{E}\boldsymbol{x}^T\mathbf{\Delta}\boldsymbol{x}$. That is

THEOREM 3.6. *The quadric* $Q \equiv \boldsymbol{x}^T\mathbf{\Delta}\boldsymbol{x} = 0$ *is invariant under all the collineations* $\mathbf{\Gamma}$.

THEOREM 3.7. *The invariant space,* γ, *of a singular collineation* $\mathbf{\Gamma}$ *lies on* Q.

4. The space of conjugates of a point. The conjugates of a point P, $(p_{\alpha\beta\gamma\delta})$, under the linear family of Clifford collineations $\mathbf{\Gamma}$ are linearly dependent on the nine points $\bar{P} = \mathbf{Y}P$, $P_\lambda^i = \mathbf{Y}_\lambda^i P$, where, e. g., P_1^2 is given by $x_{\alpha 0\beta\gamma} = 0$, $x_{\alpha 1\beta\gamma} = (-1)^{\beta+\gamma}p_{\alpha 0\beta\gamma}$. In general these points determine a [8].

Take now the eight primes ϖ_λ^i of which the equations, each involving eight terms, are

$$\varpi_\lambda^1: \quad \Sigma(-1)^{\alpha+\gamma}p_{\lambda\alpha\beta\gamma}x_{\lambda\alpha'\beta'\gamma'} = 0,$$

etc., the fixed index for ϖ_λ^i being in the i-th place, and the sign being determined in each case by $(-1)^{\alpha+\gamma}$. Using the matrices $\boldsymbol{W}$ we may write these equations as

$$\begin{aligned}
\varpi_0^1: &\quad \boldsymbol{p}^T(\boldsymbol{W}_{0111}^{0101} + \boldsymbol{W}_{0111}^{1101})\boldsymbol{x} = 0,\\
\varpi_1^1: &\quad \boldsymbol{p}^T(\boldsymbol{W}_{0111}^{0101} - \boldsymbol{W}_{0111}^{1101})\boldsymbol{x} = 0,\\
\varpi_0^2: &\quad \boldsymbol{p}^T(\boldsymbol{W}_{1011}^{1001} + \boldsymbol{W}_{1011}^{1101})\boldsymbol{x} = 0, \text{ etc.}
\end{aligned}$$

From the following relations we see that ϖ_0^1 contains $\bar{P}$, P_1^1, and the six points P_λ^i, $j = 2, 3, 4$, which may be typified by P_λ^2:

$$\begin{aligned}
\bar{P}: &\quad \boldsymbol{p}^T(\boldsymbol{W}_{0111}^{0101} + \boldsymbol{W}_{0111}^{1101})\boldsymbol{W}_{0000}^{1111}\boldsymbol{p} = -\boldsymbol{p}^T(\boldsymbol{W}_{0111}^{1010} + \boldsymbol{W}_{0111}^{0010})\boldsymbol{p}\\
&\quad = 0 \text{ since the two matrices } \boldsymbol{W} \text{ are skew-symmetric.}
\end{aligned}$$

$$\begin{aligned}
P_1^1: &\quad \boldsymbol{p}^T(\boldsymbol{W}_{0111}^{0101} + \boldsymbol{W}_{0111}^{1101})(\boldsymbol{W}_{1000}^{1111} - \boldsymbol{W}_{1000}^{0111})\boldsymbol{p}\\
&\quad = \boldsymbol{p}^T(-\boldsymbol{W}_{1111}^{1010} + \boldsymbol{W}_{1111}^{0010} - \boldsymbol{W}_{1111}^{0010} + \boldsymbol{W}_{1111}^{1010})\boldsymbol{p} = 0
\end{aligned}$$

$$P_\lambda^2: \quad \boldsymbol{p}^T(\boldsymbol{W}_{0111}^{0101}+\boldsymbol{W}_{0111}^{1101})(\boldsymbol{W}_{0100}^{0111}+\boldsymbol{W}_{0100}^{0011})\boldsymbol{p}$$
$$=\boldsymbol{p}^T(-\boldsymbol{W}_{0011}^{0010}+\boldsymbol{W}_{0011}^{0110}-\boldsymbol{W}_{0011}^{1010}+\boldsymbol{W}_{0011}^{1110})\boldsymbol{p}$$
$$=0 \text{ since each of the matrices } \boldsymbol{W} \text{ is skew-symmetric.}$$

ϖ_0^1 contains P_0^1 if

$$0=\boldsymbol{p}^T(\boldsymbol{W}_{0111}^{0101}+\boldsymbol{W}_{0111}^{1101})(\boldsymbol{W}_{1000}^{1111}+\boldsymbol{W}_{1000}^{0111})\boldsymbol{p}=-2\boldsymbol{p}^T\boldsymbol{\Delta}\boldsymbol{p}.$$

From the corresponding relations among the points $\bar{P}$, P_λ^i, and the primes ϖ_μ^j, it follows that all the conjugates of P lie in all eight primes, if P lies on $Q\equiv\boldsymbol{x}^T\boldsymbol{\Delta}\boldsymbol{x}=0$. I. e.,

THEOREM 4.1. *The necessary and sufficient condition that the space of conjugates of a point P should be of dimension less than 8 is that P should lie on the quadric Q.*

There are two systems of [7]-generators on Q, each of freedom 28, and such that a general pair of spaces of opposite systems has a common point. The invariant [7]'s γ, of the singular collineations belong to one system and have freedom 7, the [7]'s above belong to the other system and have freedom 14.

Take next the 24 primes $\varpi_{\lambda\mu}^{ij}$, typified by

$$\varpi_{\lambda\mu}^{24}: \quad \Sigma(-1)^\alpha p_{\alpha'\lambda\beta'\mu}x_{\alpha\lambda\beta\mu}=0,^4$$

the sum extending over the four terms given by possible pairs of $\alpha\beta$. These primes contain all nine points $\bar{P}$, P_λ^i, as well as P itself, if, in addition to lying on Q, the point P lies on the 8 quadric [7]-cones:

$$q_\lambda^1\equiv\Sigma(-1)^\alpha x_{\lambda\alpha\beta 0}x_{\lambda\alpha'\beta' 1}=0,$$
$$q_\lambda^2\equiv\Sigma(-1)^\alpha x_{\alpha\lambda\beta 0}x_{\alpha'\lambda\beta' 1}=0,$$
$$q_\lambda^3\equiv\Sigma(-1)^\alpha x_{\alpha\beta\lambda 0}x_{\alpha'\beta'\lambda 1}=0,$$
$$q_\lambda^4\equiv\Sigma(-1)^\alpha x_{\alpha\beta 0\lambda}x_{\alpha'\beta' 1\lambda}=0.^5$$

In terms of the matrices $\boldsymbol{W}$ these quadrics are determined by:

$$q_\lambda^1: \quad \boldsymbol{W}_{0111}^{0101}\pm\boldsymbol{W}_{0111}^{1101}$$

---

[4] These may be expressed in matrix form using the sums of four $\boldsymbol{W}$'s.

[5] Note added in proof: In addition to these quadrics the point has also to lie on Q_0+Q_1 (as defined after Theorem 4.2). This does not affect any of the theorems and involves only minor adjustments of the text.

etc., the indices $\begin{smallmatrix}0\\0\end{smallmatrix}$ and $\begin{smallmatrix}1\\0\end{smallmatrix}$ being inserted in the i-th position among $\begin{smallmatrix}101\\111\end{smallmatrix}$ for q_0^i and q_1^i respectively. That is, the nine quadrics Q, q_λ^i define a linear family of freedom 8 depending on the quadrics determining by the nine matrices $\boldsymbol{W}_{1111}^{1010}$, $\boldsymbol{W}_{0111}^{0101}$, $\boldsymbol{W}_{0111}^{1101}$, etc.

These nine do not of course themselves form a Clifford set (since none of them is skew-symmetric), but if each is multiplied by $\boldsymbol{W}_{1111}^{0101}$, we obtain the Clifford set $\boldsymbol{W}_{0000}^{1111}$, $\boldsymbol{W}_{0001}^{1111}$, $\boldsymbol{W}_{0001}^{1110}$, $\cdots$, which is derived from the primary Clifford set by effecting the interchanges $(1,4)$, $(2,3)$ among the positions of the indices. It follows that $\Phi \equiv fQ + \Sigma f_\lambda{}^i q_\lambda{}^i = 0$ is in general a non-singular quadric, but that, if $(f)^2 + \Sigma f_0{}^i f_1{}^i = 0$, it is a $[7]$-cone.

There are thus ∞^7 $[7]$-cones forming a quadric system in the linear ∞^8 system of quadrics, the vertices of the cones being the invariant spaces, γ, of the singular collineations.

Writing Φ in the form

$$\begin{aligned}\Phi &= \boldsymbol{x}^T(g\boldsymbol{W}_{1111}^{1010} + g_0^1\boldsymbol{W}_{0111}^{0101} + g_1^1\boldsymbol{W}_{0111}^{1101} + \cdots)\boldsymbol{x}\\ &= \boldsymbol{x}^T\boldsymbol{G}\boldsymbol{x}, \text{ say,}\end{aligned}$$

it may be verified that

$$\boldsymbol{x}^T\boldsymbol{\Gamma}^T\boldsymbol{G}\boldsymbol{\Gamma}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{G}^*\boldsymbol{x},$$

where the form on the right hand side is that of another quadric of the ∞^8 system. Thus the family of quadrics is invariant under the system of collineations $\boldsymbol{\Gamma}$, and the locus $\mathcal{K}$ common to the family is therefore transformed into itself by all collineations $\boldsymbol{\Gamma}$. We have then

THEOREM 4.2. *If P lies on $\mathcal{K}$, then it lies in the space, κ, of its conjugates, and κ lies in the meet of the twenty-four primes $\varpi_{\lambda\mu}^{ij}$; κ lies on $\mathcal{K}$.*

In the next section we prove that κ is of dimension 4, and $\mathcal{K}$ of dimension 10.

The nine quadratic forms, Q, $q_\lambda{}^i$, are linearly independent, but are connected by a number of relations quadratic in the coordinates. We may express these most easily by separating the form Q into two parts, thus

$$Q_0 \equiv \Sigma(-1)^\beta x_{\alpha\beta}\,\overline{\alpha+\beta}\,{}_0 x_{\alpha'\beta'}\,\overline{\alpha+\beta}'_1$$

$$Q_1 \equiv \Sigma(-1)^\beta x_{\alpha\beta}\,\overline{\alpha+\beta}'_0 x_{\alpha'\beta'}\,\overline{\alpha+\beta}\,{}_1,$$

so that $Q \equiv Q_0 - Q_1$. Then we have the 16 relations

$$\begin{aligned}x_{\alpha\beta\gamma\delta}Q_\lambda \equiv\ & x_{\alpha'\beta\gamma\delta}q_\alpha{}^1 + (-1)^\alpha x_{\alpha\beta'\gamma\delta}q_\beta{}^2\\ &+ (-1)^{\alpha+\beta}x_{\alpha\beta'\gamma\delta}q_\gamma{}^3 + (-1)^{\alpha+\beta+\gamma}x_{\alpha\beta\gamma\delta'}q_\delta{}^4.\end{aligned}$$

where $\lambda' = \alpha+\beta+\gamma+\delta$.

5. The locus of a point which lies in the space of its conjugates. If $\boldsymbol{W}^{(\alpha_i)}_{(\beta_j)}$ is taken to be a collineation matrix, then from Theorem 1.2,

$$\boldsymbol{W}^{(\alpha_i)}_{(\beta_j)} A_{(\gamma_i)} = (-1)^{\Sigma\alpha_i(\beta_i+\gamma_i)} A_{(\beta_i+\gamma_i)}.$$

So that, using the notation of § 3,

$$\boldsymbol{Y} A_{\alpha\beta\gamma\delta} = (-1)^{\alpha+\beta+\gamma+\delta} A_{\alpha\beta\gamma\delta}$$

$$\boldsymbol{Y}^1_0 A_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\{(-1)^{\alpha+\beta+\gamma+\delta} + (-1)^{\beta+\gamma+\delta}\} A_{\alpha'\beta\gamma\delta}, \text{ etc}$$

Consider now the spaces of conjugates determined by A_{0000} $(= A)$. Taking $\boldsymbol{\Gamma}_b = b\boldsymbol{Y} + \Sigma b^i_0 \boldsymbol{Y}^i_0 + \Sigma b^i_1 \boldsymbol{Y}^i_1$, we have

$$B = \boldsymbol{\Gamma}_b A = bA_{0000} + b^1_0 A_{1000} + b^2_0 A_{0100} + b^3_0 A_{0010} + b^4_0 A_{0001},$$

i. e. the space of conjugates of A_{0000} is the [4],

$\alpha = [A_{0000}, A_{1000}, A_{0100}, A_{0010}, A_{0001}]$, and A and α lie on $\mathcal{K}$. Similarly for any sets of values b, c, d, e we find that $\boldsymbol{\Gamma}_c \boldsymbol{\Gamma}_b A_{0000}$ is a set of points lying entirely in the space determined by all points $A_{\alpha\beta\gamma\delta}$ with two or fewer subscripts "1," and $\boldsymbol{\Gamma}_d \boldsymbol{\Gamma}_c \boldsymbol{\Gamma}_b A$ by all points with three or fewer subscripts "1." That is, the points $\boldsymbol{\Gamma}_c \boldsymbol{\Gamma}_b A$ lie in the [10] α^* given by $x_{1111}, x_{0111}, x_{1011}, x_{1101}, x_{1110} = 0$ and $\boldsymbol{\Gamma}_d \boldsymbol{\Gamma}_c \boldsymbol{\Gamma}_b A$ all lie in the prime, A^*, $x_{1111} = 0$. $\boldsymbol{\Gamma}_e \boldsymbol{\Gamma}_d \boldsymbol{\Gamma}_c \boldsymbol{\Gamma}_b A$ does not in general lie in any element of the simplex $\{A_{\alpha\beta\gamma\delta}\}$.

Any point B of α may be transformed into A_{0000} (by the collineation $\boldsymbol{\Gamma}_b$), so that the space β of the conjugates of B is also a [4] lying on $\mathcal{K}$. Next any point C of β may be transformed into B and thence into A_{0000}, so that the space of conjugates of C is a [4], and all these spaces lie on $\mathcal{K}$ and in the prime A^*, forming its complete intersection with $\mathcal{K}$.

Now suppose E is any other point of $\mathcal{K}$. Whatever the coordinates of E, it is clear that at least one of the points $\boldsymbol{Y}E$, $\boldsymbol{Y}^i_\lambda E$ is different from E, so that E has a line of conjugates (at least). This meets A^* in a point D, the space of conjugates of which passes through E. Thus E is also reducible to A, i. e.,

THEOREM 5.1. *There exist collineations, formed from products of the collineations* Γ, *which transform any point of* $\mathcal{K}$ *into* A_{0000}, *i. e., all points of* $\mathcal{K}$ *are projectively equivalent.*

From this it follows that all properties of $\mathcal{K}$ proved in relation to A_{0000} are valid for all points of $\mathcal{K}$.

$\mathcal{K}$ is the complete intersection of the quadrics Q_λ, q^i_λ, so that its tangent

space at any point is the meet of the tangent primes to the quadrics at the point. From this we find, in particular

THEOREM 5.2. *The* [10] α^* *is the tangent prime to* $\mathcal{K}$ *at* A, *and therefore*

THEOREM 5.3. $\mathcal{K}$ *is of dimension* 10.

THEOREM 5.4. *The tangent-*[10]*'s at all points of* α *lie in the prime* A^*.

The invariant-[7]'s, γ (Theorem 3.6), all lie on Q. Particular among them are the invariant spaces of the collineations $\mathbf{Y}^i_\lambda$. Using the form given at the beginning of this section we find that the invariant space of $\mathbf{Y}^1_0$ is the space determined by the eight points $A_{0\beta\gamma\delta}$. This lies on all quadrics q^i_λ except q^1_0, which it cuts in the non-singular quadric sixfold

$$\tilde{q}^1_0 = \Sigma(-1)^{\alpha} x_{0\alpha\beta 0} x_{0\alpha'\beta' 1} = 0,$$

(with $x_{1\beta\gamma\delta} = 0$).

THEOREM 5.5. *There are* ∞^7 [7]*'s,* γ, *each cutting* $\mathcal{K}$ *in a quadric sixfold.*

I prove next

THEOREM 5.6. *The locus of space* κ_A *which pass through* A *is a cone* $A\mathcal{G}$ *projecting the Grassmannian* $\mathcal{G}^5_6$ *of lines of* [4].

The required locus is the section of $\mathcal{K}$ by α^*; α^* lies on the four quadrics q^i_1, and meets the other quadrics in:

$$\begin{aligned}
q^1_0 &: \quad -x_{0100}x_{0011} + x_{0010}x_{0101} - x_{0110}x_{0001} = 0\\
q^2_0 &: -x_{1000}x_{0011} + x_{0010}x_{1001} - x_{1010}x_{0001} = 0\\
q^3_0 &: -x_{1000}x_{0101} + x_{0100}x_{1001} - x_{1100}x_{0001} = 0\\
q^4_0 &: -x_{1000}x_{0110} + x_{0100}x_{1010} - x_{1100}x_{0010} = 0\\
Q &: -x_{0110}x_{1001} + x_{1010}x_{0101} - x_{1100}x_{0011} = 0.
\end{aligned}$$

In place of $x_{\alpha_1\alpha_2\alpha_3\alpha_4}$, write $y_{ij}, = y_{ji}$, when $\alpha_i = \alpha_j = 1$ and the other two subscripts are zero, and write $y_{i0}, = y_{0i}$, when $\alpha_i = 1$ and the other three are zero. These equations are then usual equations of the Grassmannian in [9] of lines in [4], except that the signs are written into the equations instead of being carried by the convention $z_{ij} + z_{ji} = 0$.

Properties of $\mathcal{G}$ are most easily discussed in relation to figures in [4]; take then a [4] and name its elements π-points, π-lines, etc. On $\mathcal{G}$ there

is a single ∞^4 system of solids g (each corresponding to π-lines through a π-point), each pair with a single common point (corresponding to the π-line joining two π-points). To the lines in a π-plane there correspond points of a plane s on $\mathfrak{G}$: through any line on $\mathfrak{G}$ there passes on plane s and one solid g (the line corresponds to a pencil of π-lines of which the vertex is represented by g and the plane by s). Through any point of $\mathfrak{G}$ there are ∞^1 solids g and ∞^2 planes s forming a cone of order 3, the complete section by the tangent-[6].[6]

Thus for the cone $A\mathfrak{G}$ we find:

THEOREM 5.7. *On $A\mathfrak{G}$ in the* [10] α^* *there are* ∞^4 [4]*'s* κ_A *all through A, and each pair with a common line, also* ∞^6 [3]*'s* σ, *all through A. Through any plane through A and on $A\mathfrak{G}$ there pass one* κ *and one* σ.

A "typical" σ is $[A_{0000}, A_{0100}, A_{0010}, A_{0110}]$.

THEOREM 5.8. *The spaces of conjugates of points of a line through A and in α all pass through the line and form a cubic line-cone* (| 2, 3 |, [7]).

Associated with $\mathfrak{G}$ there are ∞^4 [5]'s (corresponding to π-solids) one through each point of the [9], and each cutting $\mathfrak{G}$ in a quadric fourfold, the two systems of generators on which are planes s and section of solids g. Each of these quadrics gives a quadric point-cone in [6] on $A\mathfrak{G}$, with generators σ and sections of κ. This point-cone is the section by a tangent prime of the quadric $\bar{q}$ in one of the invariant spaces γ.

In Theorem 5.7 we proved that each [4] κ_A through A meets each other κ_A in a line and no more. Now take B in α, and β the space of its conjugates. α, β meet only in the line AB. Each space κ_B which passes through B meets every other κ_B in a line through B. But amongst the κ_B is α, so that any two [4]'s κ meet in a line (exactly) or not at all, and all the κ's that meet any [4] α of the family in lines lie in the tangent prime A^*.

We may summarize the structure of $\mathfrak{K}$ thus:

THEOREM 5.9. *There are on $\mathfrak{K}$*: (i) ∞^{10} [4]*'s* κ *two of which in general do not meet but each meets* ∞^6 *other in lines,* (ii) ∞^{13} [3]*'s* σ, *any pair of which either do not meet, or they may have a point or a line common. Through any plane of $\mathfrak{K}$ there pass one* κ *and one* σ, (iii) ∞^7 *quadric sixfolds $\bar{q}$ on which the generators are solids* σ *and sections of* [4]*'s* κ.

[6] This cone is the "determinantal locus" (| 2, 3 |, [6]). See, e.g., Room (1938), p. 202.

Let us return now to the systems of:

points K, [4]'s κ, [10]'s κ^*, and primes K^* each of freedom 10.

Any set of sixteen homogeneous parameters $\xi_{\alpha\beta\gamma\delta}$, satisfying the ten relations $Q_\lambda = 0$, $q_\lambda^t = 0$, determines: the point K, of which they are the coordinates,
the space κ of its conjugates (from the primes $\varpi_{\lambda\mu}^{tj}$, § 4),
the tangent-[10]'s κ^* to $\mathcal{K}$ at K,
the prime K^* containing the tangent-[10]'s at all points of κ.

Any collineation Γ (or product of Γ's) replaces all these by other spaces of the same system. Any correlation Γ (or product of a correlation Γ with collineations Γ) replaces each K by a K^* and each κ by a κ^*.

Thus any incidence relation among a set of points K is reproduced by the corresponding primes K^* and likewise of κ by κ^*. If we can show that there are incidence relations among a set of spaces κ corresponding to given incidence relations among a set of points K, we shall have established a complete "*mutuality*" among all four systems. In general terms the existence of such a correspondence is clear, because each κ is the space of conjugates of a definite point K of itself. The detail of some of these correspondences is:

(i) Two points K_1, K_2 of a space κ determine a line lying in κ; two spaces κ_1, κ_2 which meet in a line determine an ∞^1 system through the line forming $(|\,2, 3\,|, [7])$.

(ii) To points of a solid σ correspond the [4]'s κ meeting it in planes.

(iii) On any quadric sixfold $\tilde{q}$ there is the Study "triality" linking the points, with the spaces κ the sections of which are one system of generators, and the spaces κ^* the sections of which are the tangent-[6]'s to $\tilde{q}$.

We may therefore say

THEOREM 5.10. *There is a "mutuality" among the systems of spaces K, κ, κ^*, K^* analogous to Study's "triality" of points, generator-*[3]*'s, and tangent primes of a quadric in* [7].

In the general configuration in $[2^r - 1]$, we will have a "mutuality" among systems of spaces of dimensions $\nu_0, \nu_1, \nu_2, \cdots, \nu_{r-1}$ where

$$\nu_0 = 0, \nu_1 = r, \cdots, \nu_i = \binom{r}{1} + \binom{r}{2} + \cdots + \binom{r}{i}, \nu_{r-1} = [2^r - 2].^7$$

[7] For this sequence of numbers, see for example, Room (1936), p. 539.

Part III.

Two generalizations of the commutative-anti-commutative set.

6. Sets, in general commutative, but with some pairs for which $VU = kUV$. The matrices to be considered have n^r rows and columns. Number the rows and columns in the reversed scale of n, so that $\alpha_1\alpha_2 \cdots \alpha_r$ is the index of the row or column with position number $\alpha_1 + n\alpha_2 + \cdots + n^{r-1}a_r$. I make the conventions that:

(i) Greek indices takes the values $0, 1, \cdots, n-1$, and Italic indices take the values $1, \cdots, r$, and

(ii) all arithmetical functions of the Greek indices are reduced modulo n.

Denote the elements of a matrix by $a^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_r}$, or $a^{(\alpha_i)}_{(\beta_j)}$, and define matrices $\boldsymbol{U}_i$, $\boldsymbol{V}_i$ thus:

$\boldsymbol{U}_i$: the diagonal matrix with elements $a^{\alpha_1\cdots\alpha_r}_{\alpha_1\cdots\alpha_r} = \epsilon^{\alpha_i}$, where

$$\epsilon = \exp(2\pi i/n).$$

$\boldsymbol{V}_i$: the substitution matrix, with one non-zero element in each row and column, given by

$$a^{\alpha_1\alpha_2\cdots\alpha_i\cdots\alpha_r}_{\alpha_1\alpha_2\cdots\alpha_i+1\cdots\alpha_r} = 1,$$

where each subscript except the i-th is equal to the corresponding superscript, while the i-th subscript exceeds by 1 the i-th superscript. For $n = 3$, $r = 2$, the matrices are:

$$\boldsymbol{U}_1 = ((1\epsilon\epsilon^2 1\epsilon\epsilon^2)), \quad \boldsymbol{U}_2 = ((111\epsilon\epsilon\epsilon\ \epsilon^2\epsilon^2\epsilon^2)), \text{ where } \epsilon^2 + \epsilon + 1 = 0.$$

$\boldsymbol{V}_1$, $\boldsymbol{V}_2$ correspond respectively to the transpositions 231564897, 456789123.

For these matrices we may prove:

THEOREM 6.1. (i) $\boldsymbol{U}_i^n = \boldsymbol{V}_i^n = I$.

(ii) *Every pair of matrices commutes, except the r pairs, $\boldsymbol{U}_i\boldsymbol{V}_i$.*

(iii) $\boldsymbol{V}_i\boldsymbol{U}_i = \epsilon\boldsymbol{U}_i\boldsymbol{V}_i$.

THEOREM 6.2. $\boldsymbol{W}^{(\alpha_i)}_{(\beta_j)} = \Pi\boldsymbol{U}_i^{\alpha_i}\,\Pi\boldsymbol{V}_j^{\beta_j}$, *is the matrix with one non-zero element in each row and column, given by* $a^{(\lambda_i)}_{(\lambda_j+\beta_j)} = \epsilon^{\Sigma\alpha_i\lambda_i}$.

THEOREM 6.3. $\boldsymbol{W}^{(\alpha_i)}_{(\beta_j)}\boldsymbol{W}^{(\gamma_i)}_{(\delta_j)} = \epsilon^{\Sigma\beta_i\lambda_i}\boldsymbol{W}^{(\alpha_i+\gamma_i)}_{(\beta_j+\delta_j)}$.

THEOREM 6.4. $[\boldsymbol{W}^{(\alpha_i)}_{(\beta_j)}]^n = \begin{cases} \boldsymbol{I} \text{ if } n \text{ is odd,} \\ (-1)\epsilon^{\Sigma\alpha_i\beta_i}\boldsymbol{I} \text{ if } n \text{ is even.} \end{cases}$

THEOREM 6.5. *The n^{2r+1} matrices $\epsilon^{\sigma} W_{(\beta_j)}^{(\alpha_i)}$ form a group. The n^{2r} collineations determined by them form a group.*

THEOREM 6.6. *Any matrix of n^r rows and columns may be expressed in a unique way as $\Sigma k_{(\delta_j)}^{(\gamma_i)} W_{(\delta_j)}^{(\gamma_i)}$, there being n^{2r} terms in the summation.*

Thus, let $\boldsymbol{H} = (h_{(\beta_j)}^{(\alpha_i)})$ be an arbitrary matrix, and let $\tilde{\boldsymbol{H}} = (\tilde{h}_{(\beta_j)}^{(\alpha_i)})$ be the matrix obtained by rearranging the terms in each row in the way determined by

$$h_{(\beta_i)}^{(\alpha_i)} = h_{(\alpha_i+\beta_i)}^{(\alpha_i)}.$$

Then we may prove that

$$\tilde{\boldsymbol{H}} = \boldsymbol{\Omega K}$$

where $\boldsymbol{K} = (k_{(\beta_j)}^{(\alpha_i)})$, and $\boldsymbol{\Omega}$ is the matrix the elements of which are the powers of ϵ given by

$$a_{(\beta_i)}^{(\alpha_i)} = \epsilon^{\Sigma \alpha_i \beta_i}.$$

For example, for $n = 3$, $r = 2$, we have

$$\tilde{\boldsymbol{H}} = \begin{pmatrix} h_{00}^{00} & h_{10}^{00} & h_{20}^{00} & h_{01}^{00} & h_{11}^{00} & h_{21}^{00} & h_{02}^{00} & h_{12}^{00} & h_{22}^{00} \\ h_{10}^{10} & h_{20}^{10} & h_{00}^{10} & h_{11}^{10} & h_{21}^{10} & h_{01}^{10} & h_{12}^{10} & h_{22}^{10} & h_{02}^{10} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{22}^{22} & h_{02}^{22} & h_{12}^{22} & h_{20}^{22} & h_{00}^{22} & h_{10}^{22} & h_{21}^{22} & h_{01}^{22} & h_{11}^{22} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 & 1 & \epsilon & \epsilon^2 & 1 & \epsilon & \epsilon^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \epsilon^2 & \epsilon & \epsilon^2 & \epsilon & 1 & \epsilon & 1 & \epsilon^2 \end{pmatrix} \begin{pmatrix} k_{00}^{00} & k_{10}^{00} & k_{20}^{00} & k_{01}^{00} & \cdots & k_{22}^{00} \\ k_{00}^{10} & k_{10}^{10} & k_{20}^{10} & k_{01}^{10} & \cdots & k_{22}^{10} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{00}^{22} & k_{10}^{22} & k_{20}^{22} & k_{00}^{22} & \cdots & k_{22}^{22} \end{pmatrix}.$$

The problems that these matrices suggest immediately are

(i) to pick out Clifford sets of matrices $\boldsymbol{W}$, namely sets no two of whose members commute,

(ii) to pick out sets $\boldsymbol{W}_R$, such that $(\Sigma c_R \boldsymbol{W}_R)^n = f(c_R)\boldsymbol{I}$,

(iii) to find geometrical interpretations of the collineation group determined by the family of matrices.

7. Sets of matrices, in general commutative, but with some pairs for which $\boldsymbol{VU} = \boldsymbol{U}^2\boldsymbol{V}$. Number the rows and columns of a $n^r \times n^r$ matrix as in § 6, and take as basic matrices:

$\boldsymbol{U}_i$: the diagonal matrix

$$a\,{}^{\alpha_1\ldots\alpha_r}_{\alpha_1\ldots\alpha_r} = \eta^{2\alpha_i}$$

where

$$\eta = \exp\{2\pi i/(2^n - 1)\}.$$

$\boldsymbol{V}_i$: the substitution matrix defined in § 6.

For $n = 3$, $r = 2$, we have:

$$\boldsymbol{U}_1 = ((\eta\ \eta^2\ \eta^4\ \eta\ \eta^2\ \eta^4\ \eta\ \eta^2\ \eta^4)),$$

$$\boldsymbol{U}_2 = ((\eta\ \eta\ \eta\ \eta^2\ \eta^2\ \eta^2\ \eta^4\ \eta^4\ \eta^4)),$$

where $\eta^7 = 1$.

Then:

THEOREM 7.1. (i) $\boldsymbol{U}_i^{2^n-1} = \boldsymbol{V}_i^n = \boldsymbol{I}$.

(ii) *Every pair of matrices commutes, except the r pairs* $\boldsymbol{U}_i$, $\boldsymbol{V}_i$.

(iii) $\boldsymbol{V}_i\boldsymbol{U}_i = \boldsymbol{U}_i^2\boldsymbol{V}_i$.

THEOREM 7.2. $\boldsymbol{V}_i^\beta \boldsymbol{U}_i^\alpha = \boldsymbol{U}_i^{2^\beta\alpha}\boldsymbol{V}_i^\beta$.

THEOREM 7.3. *If* $\boldsymbol{W}\,{}^{(\alpha_i)}_{(\beta_j)} = \Pi \boldsymbol{U}_i^{\alpha_i}\Pi \boldsymbol{V}_j^{\beta_j}$, *then*

$$\boldsymbol{W}\,{}^{(\alpha_i)}_{(\beta_j)}\,\boldsymbol{W}\,{}^{(\gamma_i)}_{(\delta_j)} = \boldsymbol{W}\,{}^{(\alpha_i+2^{\beta_i}\gamma_i)}_{(\beta_j+\delta_j)},$$

where the superscripts of $\boldsymbol{W}$ *are reduced modulo* $2^n - 1$, *and the subscripts modulo n.*

THEOREM 7.4. *The matrices* $\boldsymbol{W}$ *form a group in which*

$$[\boldsymbol{W}\,{}^{(\alpha_i)}_{(\beta_i)}]^{-1} = \boldsymbol{W}\,{}^{(-2^{n-\beta_i}\alpha_i)}_{(n-\beta_i)}.$$

8. Further generaliaztions. Let $n_1, n_2, \cdots, n_r$ be any r positive integers, not necessarily all different, each greater than 1, and let $N = \Pi n_i$, $i = 1, \cdots, r$. Write the position numbers, beginning with 0, of the rows and columns of a N-by-N matrix in the form

$$\alpha_1 + n_1\alpha_2 + n_1n_2\alpha_3 + \cdots + n_1n_2\cdots n_{r-1}\alpha_r,$$

where α_i is to take the values $0, 1, \cdots, n_i - 1$. Take $a\,{}^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_r}$ or $a\,{}^{(\alpha_i)}_{(\beta_j)}$ as the elements of a matrix. Then construct the two sets of r matrices:

$\boldsymbol{U}_i$: the diagonal matrix in which

$$a\,{}^{\alpha_1\ldots\alpha_r}_{\alpha_1\ldots\alpha_r} = \epsilon^{\alpha_i}, \text{ where } \epsilon_i = \exp(2i\pi/n_i)$$

$\boldsymbol{V}_i$: the substitution matrix with non-zero elements $a\,{}^{\alpha_1\ldots\alpha_i\ldots\alpha_r}_{\alpha_1\ldots\alpha_i+1\ldots\alpha_r} = 1$, each subscript except the i-th being equal to the corresponding superscript. E. g., for $n_1 = 2$, $n_2 = 3$, $n_3 = 2$ the matrices are

$$U_1 = ((1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1)),$$

$$U_2 = ((1 \quad 1 \quad \epsilon \quad \epsilon \quad \epsilon^2 \quad \epsilon^2 \quad 1 \quad 1 \quad \epsilon \quad \epsilon \quad \epsilon^2 \quad \epsilon^2)),$$

where $\epsilon^2 + \epsilon + 1 = 0$,

$$U_3 = ((1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1)).$$

V_1, V_2, V_3 correspond to the substitutions: 1032547698*et*, 23450189*te*67, 6789*te*012345.

These matrices have properties analogous to those discussed in § 6, the difference being that the number n_i involved is different for each pair of matrices U_i, V_i, instead of being the same for all pairs.

THEOREM 8.1. (i) $U_i^{n_i} = I$, $V_i^{n_i} = I$.

(ii) *Every pair of matrices commutes except the r pairs* U_i, V_i.

(iii) $V_i U_i = \epsilon_i U_i V_i$, *where* $\epsilon_i = \exp(2i\pi/n_i)$.

THEOREM 8.2. *If* $W_{\beta_1 \ldots \beta_r}^{\alpha_1 \ldots \alpha_r} = \Pi U_i^{\alpha_i} \Pi V_i^{\beta_i}$, *then*

$$W_{(\beta_i)}^{(\alpha_i)} W_{(\delta_i)}^{(\gamma_i)} = (\Pi \epsilon_i^{\beta_i \gamma_i}) W_{(\beta_i + \delta_i)}^{(\alpha_i + \gamma_i)},$$

where the i-th index is reduced modulo n_i.

THEOREM 8.3. *Any N-by-N matrix may be expressed in a single way in the form* $\Sigma k_{(\beta_i)}^{(\alpha_i)} W_{(\beta_i)}^{(\alpha_i)}$ *for a given ordered set of factors* $n_1, n_2, \cdots, n_r$ *of* N.

Thus, if $H = (h_{(\beta_i)}^{(\alpha_i)})$ is the matrix, then, writing

$$\tilde{H} = (\tilde{h}_{(\beta_i)}^{(\alpha_i)}) \text{ where } \tilde{h}_{(\beta_i)}^{(\alpha_i)} = h_{(\alpha_i + \beta_i)}^{(\alpha_i)}, \qquad K = (k_{(\beta_i)}^{(\alpha_i)}),$$

$$\Omega = \text{the matrix with elements } a_{\beta_1 \beta_2 \ldots \beta_r}^{\alpha_1 \alpha_2 \ldots \alpha_r} = \Pi \epsilon_i^{\alpha_i \beta_i},$$

we have $\tilde{H} = \Omega K$.

E. g., for $n_1 = 3$, $n_2 = 2$:

$$\begin{pmatrix} h_{00}^{00} & h_{10}^{00} & h_{20}^{00} & h_{01}^{00} & h_{11}^{00} & h_{21}^{00} \\ h_{10}^{10} & h_{20}^{10} & h_{00}^{10} & h_{11}^{10} & h_{21}^{10} & h_{01}^{10} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{21}^{21} & h_{01}^{21} & h_{11}^{21} & h_{20}^{21} & h_{00}^{21} & h_{10}^{21} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 & 1 & \epsilon & \epsilon^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \epsilon^2 & \epsilon & -1 & -\epsilon^2 & -\epsilon \end{pmatrix} \begin{pmatrix} k_{00}^{00} & k_{10}^{00} & \cdots & k_{21}^{00} \\ k_{00}^{10} & k_{10}^{10} & \cdots & k_{21}^{10} \\ \cdots & \cdots & \cdots & \cdots \\ k_{00}^{21} & k_{10}^{21} & \cdots & k_{21}^{21} \end{pmatrix}.$$

The family of matrices discussed in § 7 may be generalized in the same way, and may be generalized also by defining the η_i in terms of powers which are powers of numbers t_i different from 2, so as to obtain relations $V_iU_i = U_i^{t_i}V_i$.

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